

BOOLEAN VALUED ANALYSIS OF ORDER BOUNDED OPERATORS

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Abstract

This is a survey of some recent applications of Boolean valued models of set theory to order bounded operators in vector lattices.

Key words: Boolean valued model, transfer principle, descent, ascent, order bounded operator, disjointness, band preserving operator, Maharam operator.

Introduction

The term *Boolean valued analysis* signifies the technique of studying properties of an arbitrary mathematical object by comparison between its representations in two different set-theoretic models whose construction utilizes principally distinct Boolean algebras. As these models, we usually take the classical Cantorian paradise in the shape of the von Neumann universe and a specially-trimmed Boolean valued universe in which the conventional set-theoretic concepts and propositions acquire bizarre interpretations. Use of two models for studying a single object is a family feature of the so-called *nonstandard methods of analysis*. For this reason, Boolean valued analysis means an instance of nonstandard analysis in common parlance.

Proliferation of Boolean valued analysis stems from the celebrated achievement of P. J. Cohen who proved in the beginning of the 1960s that the negation of the continuum hypothesis, CH, is consistent with the axioms of Zermelo–Fraenkel set theory, ZFC. This result by Cohen, together with the consistency of CH with ZFC established earlier by K. Gödel, proves that CH is independent of the conventional axioms of ZFC.

The first application of Boolean valued models to functional analysis were given by E. I. Gordon for Dedekind complete vector lattices and positive operators in [22]–[24] and G. Takeuti self-adjoint operators in Hilbert spaces and harmonic analysis in [76]–[78]. The further developments and corresponding references are presented in [48, 49].

The aim of the paper is to survey some recent applications of Boolean valued models of set theory to studying order bounded operators in vector lattices. Chapter 1 contains a sketch of the adaptation to analysis of the main constructions and principles of Boolean valued models of set theory. The three subsequent chapters treat the classes of operators in vector lattices: multiplication type operators, weighted shift type operators, and conditional expectation type operators.

The reader can find the necessary information on Boolean algebras in [73, 80], on the theory of vector lattices, in [10, 35, 41, 81, 84], on Boolean valued models of set theory, in [11, 33, 79], and on Boolean valued analysis, in [47, 48, 49].

Everywhere below \mathbb{B} denotes a complete Boolean algebra, while $\mathbb{V}^{(\mathbb{B})}$ stands for the corresponding Boolean valued universe (the universe of \mathbb{B} -valued sets). A *partition of unity* in \mathbb{B} is a family $(b_\xi)_{\xi \in \Xi} \subset \mathbb{B}$ with $\bigvee_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $b_\xi \wedge b_\eta = \mathbb{0}$ for $\xi \neq \eta$.

By a vector lattice throughout the sequel we will mean a real Archimedean vector lattice, unless specified otherwise. We let $:=$ denote the assignment by definition, while \mathbb{N} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} symbolize the naturals, the rationals, the reals, and the complexes. We denote the Boolean algebras of bands and band projections in a vector lattice X by $\mathbb{B}(X)$ and $\mathbb{P}(X)$; and we let X^u stand for the universal completion of a vector lattice X .

The ideal center $\mathcal{Z}(X)$ of a vector lattice X is an f -algebra. Let $\text{Orth}(X)$ and $\text{Orth}^\infty(X)$ stand for the f -algebras of orthomorphisms and extended orthomorphisms, respectively X . Then $\mathcal{Z}(X) \subset \text{Orth}(X) \subset \text{Orth}^\infty(X)$. The space of all order bounded linear operators from X to Y is denoted by $L^\sim(X, Y)$. The Riesz–Kantorovich Theorem tells us that if Y is a Dedekind complete vector lattice then so is $L^\sim(X, Y)$.

Chapter 1. Boolean Valued Analysis

1.1. Boolean Valued Models

We start with recalling some auxiliary facts about the construction and treatment of Boolean valued models. Some more detailed presentation can be found in [11, 48, 49]. In the sequel $\text{ZFC} := \text{ZF} + \text{AC}$, where ZF stands for the Zermelo–Fraenkel set theory and AC for the axiom of choice.

1.1.1. Let \mathbb{B} be a complete Boolean algebra. Given an ordinal α , put

$$\mathbb{V}_\alpha^{(\mathbb{B})} := \{ x : x \text{ is a function} \wedge (\exists \beta)(\beta < \alpha \wedge \text{dom}(x) \subset \mathbb{V}_\beta^{(\mathbb{B})} \wedge \text{Im}(x) \subset \mathbb{B}) \}.$$

After this recursive definition the *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ or, in other words, the *class of \mathbb{B} -sets* is introduced by

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \text{On}} \mathbb{V}_\alpha^{(\mathbb{B})},$$

with On standing for the class of all ordinals.

In case of the two-element Boolean algebra $\mathbb{2} := \{0, 1\}$ this procedure yields a version of the classical *von Neumann universe* \mathbb{V} where $\mathbb{V}_0 := \emptyset$, $\mathbb{V}_{\alpha+1} := \mathcal{P}(\mathbb{V}_\alpha)$, $\mathbb{V}_\beta := \bigcup_{\alpha < \beta} \mathbb{V}_\alpha$, β is a limit ordinal (cp. [49, Theorem 4.2.8]).

1.1.2. Let φ be an arbitrary formula of ZFC, Zermelo–Fraenkel set theory with choice. The *Boolean truth value* $\llbracket \varphi \rrbracket \in \mathbb{B}$ is introduced by induction on the complexity of φ by naturally interpreting the propositional connectives and quantifiers in the Boolean algebra \mathbb{B} (for instance, $\llbracket \varphi_1 \vee \varphi_2 \rrbracket := \llbracket \varphi_1 \rrbracket \vee \llbracket \varphi_2 \rrbracket$) and taking into consideration the way in which a formula is built up from atomic formulas. The Boolean truth values of the *atomic formulas* $x \in y$ and $x = y$

(with x, y assumed to be the elements of $\mathbb{V}^{(\mathbb{B})}$) are defined by means of the following recursion schema:

$$\begin{aligned}\llbracket x \in y \rrbracket &= \bigvee_{t \in \text{dom}(y)} (y(t) \wedge \llbracket t = x \rrbracket), \\ \llbracket x = y \rrbracket &= \bigvee_{t \in \text{dom}(x)} (x(t) \Rightarrow \llbracket t \in y \rrbracket) \wedge \bigvee_{t \in \text{dom}(y)} (y(t) \Rightarrow \llbracket t \in x \rrbracket).\end{aligned}$$

The sign \Rightarrow symbolizes the implication in \mathbb{B} ; i.e., $(a \Rightarrow b) := (a^* \vee b)$, where a^* is as usual the *complement* of a . The universe $\mathbb{V}^{(\mathbb{B})}$ with the Boolean truth value of a formula is a model of set theory in the sense that the following is fulfilled:

1.1.3. Transfer Principle. *For every theorem φ of ZFC, we have $\llbracket \varphi \rrbracket = \mathbb{1}$ (also in ZFC); i.e., φ is true inside the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$.*

We enter into the next agreement: If $\varphi(x)$ is a formula of ZFC then, on assuming x to be an element of $\mathbb{V}^{(\mathbb{B})}$, the phrase “ x satisfies φ inside $\mathbb{V}^{(\mathbb{B})}$ ” or, briefly, “ $\varphi(x)$ is true inside $\mathbb{V}^{(\mathbb{B})}$ ” means that $\llbracket \varphi(x) \rrbracket = \mathbb{1}$. This is sometimes written as $\mathbb{V}^{(\mathbb{B})} \models \varphi(x)$.

1.1.4. There is a natural equivalence relation $x \sim y \iff \llbracket x = y \rrbracket = \mathbb{1}$ in the class $\mathbb{V}^{(\mathbb{B})}$. Choosing a representative of the least rank in each equivalence class or, more exactly, using the so-called “Frege–Russell–Scott trick,” we obtain a *separated Boolean valued universe* $\overline{\mathbb{V}}^{(\mathbb{B})}$ for which $x = y \iff \llbracket x = y \rrbracket = \mathbb{1}$. It is easily to see that the Boolean truth value of a formula remains unaltered if we replace in it each element of $\mathbb{V}^{(\mathbb{B})}$ by one of its equivalents; cp. [49, § 4.5]. In this connection from now on we take $\mathbb{V}^{(\mathbb{B})} := \overline{\mathbb{V}}^{(\mathbb{B})}$ without further specification.

1.1.5. Given $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$, define the function $bx : z \mapsto b \wedge x(z)$ ($z \in \text{dom}(x)$). Here we presume that $b\emptyset := \emptyset$ for all $b \in \mathbb{B}$. Observe that in $\mathbb{V}^{(\mathbb{B})}$ the element bx is defined correctly for $x \in \mathbb{V}^{(\mathbb{B})}$ and $b \in \mathbb{B}$; cp. [49, § 4.3].

1.1.6. Mixing Principle. *Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in \mathbb{B} , i.e., $\sup_{\xi \in \Xi} b_\xi = \mathbb{1}$ and $\xi \neq \eta \implies b_\xi \wedge b_\eta = \emptyset$. To each family $(x_\xi)_{\xi \in \Xi}$ in $\mathbb{V}^{(\mathbb{B})}$ there exists a unique element x in $\mathbb{V}^{(\mathbb{B})}$ such that $\llbracket x = x_\xi \rrbracket \geq b_\xi$ for all $\xi \in \Xi$.*

This x is the *mixing* of $(x_\xi)_{\xi \in \Xi}$ by $(b_\xi)_{\xi \in \Xi}$ denoted by $\text{mix}_{\xi \in \Xi} b_\xi x_\xi$.

1.1.7. Maximum Principle. *Let $\varphi(x)$ be a formula of ZFC. Then (in ZFC) there is a \mathbb{B} valued set x_0 satisfying $\llbracket (\exists x) \varphi(x) \rrbracket = \llbracket \varphi(x_0) \rrbracket$.*

In particular, if it is true within $\mathbb{V}^{(\mathbb{B})}$ that “there is an x for which $\varphi(x)$ ”, then there is an element x_0 in $\mathbb{V}^{(\mathbb{B})}$ (in the sense of \mathbb{V}) with $\llbracket \varphi(x_0) \rrbracket = \mathbb{1}$. In symbols, $\mathbb{V}^{(\mathbb{B})} \models (\exists x) \varphi(x) \implies (\exists x_0) \mathbb{V}^{(\mathbb{B})} \models \varphi(x_0)$.

1.2. Escher Rules

Now, we present a remarkable interplay between \mathbb{V} and $\mathbb{V}^{(\mathbb{B})}$ which is based on the operations of canonical embedding, descent, and ascent.

1.2.1. We start with the canonical embedding of the von Neumann universe into the Boolean valued universe. Given $x \in \mathbb{V}$, we denote by x^\wedge the *standard name* of x in $\mathbb{V}^{(\mathbb{B})}$; i.e., the element defined by the following recursion schema:

$$\emptyset^\wedge := \emptyset, \quad \text{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \text{im}(x^\wedge) := \{\mathbb{1}\}.$$

Henceforth, working in the separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$, we agree to preserve the symbol x^\wedge for the distinguished element of the class corresponding to x . The map $x \mapsto x^\wedge$ is called *canonical embedding*.

A formula is *bounded* or *restricted* provided that each bound variable in it is restricted by a bounded quantifier; i.e., a quantifier ranging over a particular set. The latter means that each bound variable x is restricted by a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$.

1.2.2. Restricted Transfer Principle. *Let $\varphi(x_1, \dots, x_n)$ be a bounded formula of ZFC. Then (in ZFC) for every collection $x_1, \dots, x_n \in \mathbb{V}$ we have*

$$\varphi(x_1, \dots, x_n) \iff \mathbb{V}^{(\mathbb{B})} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

1.2.3. Given an arbitrary element x of the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, define the class $x\downarrow$ by

$$x\downarrow := \{y \in \mathbb{V}^{(\mathbb{B})} : \llbracket y \in x \rrbracket = \mathbb{1}\}.$$

This class is called the *descent* of x . Moreover $x\downarrow$ is a set, i.e., $x\downarrow \in \mathbb{V}$ for every $x \in \mathbb{V}^{(\mathbb{B})}$. If $\llbracket x \neq \emptyset \rrbracket = \mathbb{1}$, then $x\downarrow$ is a non-empty set.

1.2.4. Suppose that f is a map from X in Y within $\mathbb{V}^{(\mathbb{B})}$. More precisely, f , X and Y are in $\mathbb{V}^{(\mathbb{B})}$ and $\llbracket f : X \rightarrow Y \rrbracket = \mathbb{1}$. There exist a unique map $f\downarrow$ from $X\downarrow$ in $Y\downarrow$ (in the sense of the von Neumann universe \mathbb{V}) such that

$$\llbracket f\downarrow(x) = f(x) \rrbracket = \mathbb{1} \quad (x \in X\downarrow).$$

Moreover, for a nonempty subset A of X within $\mathbb{V}^{(\mathbb{B})}$ (i. e. $\llbracket \emptyset \neq A \subset X \rrbracket = \mathbb{1}$) we have $f\downarrow(A\downarrow) = f(A)\downarrow$. The map $f\downarrow$ from $X\downarrow$ to $Y\downarrow$ is called the *descent* of f from $\mathbb{V}^{(\mathbb{B})}$. The descent $f\downarrow$ of an internal map f is *extensional*:

$$\llbracket x = x' \rrbracket \leq \llbracket f\downarrow(x) = f\downarrow(x') \rrbracket \quad (x, x' \in X\downarrow).$$

For the descents of the composite, inverse, and identity maps we have:

$$(g \circ f)\downarrow = g\downarrow \circ f\downarrow, \quad (f^{-1})\downarrow = (f\downarrow)^{-1}, \quad (I_X)\downarrow = I_{X\downarrow}.$$

By virtue of these rules we can consider the descent operation as a functor from the category of \mathbb{B} -valued sets and mappings to the category of the standard sets and mappings (i.e., those in the sense of \mathbb{V}).

1.2.5. Given $x_1, \dots, x_n \in \mathbb{V}^{(\mathbb{B})}$, denote by $(x_1, \dots, x_n)^\mathbb{B}$ the corresponding ordered n -tuple inside $\mathbb{V}^{(\mathbb{B})}$. Assume that P is an n -ary relation on X inside $\mathbb{V}^{(\mathbb{B})}$; i.e., $\llbracket P \subset X^n \rrbracket = \mathbb{1}$ and $\llbracket P \subset X^n \rrbracket = \mathbb{1}$. Then there exists an n -ary relation P' on $X\downarrow$ such that $(x_1, \dots, x_n) \in P' \iff \llbracket (x_1, \dots, x_n)^\mathbb{B} \in P \rrbracket = \mathbb{1}$. Slightly abusing notation, we denote P' by the occupied symbol $P\downarrow$ and call $P\downarrow$ the *descent* of P .

1.2.6. Let $x \in \mathbb{V}$ and $x \subset \mathbb{V}^{(\mathbb{B})}$; i.e., let x be some set composed of \mathbb{B} -valued sets or, symbolically, $x \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$. Put $\emptyset\uparrow := \emptyset$ and $\text{dom}(x\uparrow) := x$, $\text{Im}(x\uparrow) := \{\mathbb{1}\}$ if $x \neq \emptyset$. The element $x\uparrow$ (of the non-separated universe $\overline{\mathbb{V}}^{(\mathbb{B})}$,

i. e., the distinguished representative of the class $\{y \in \overline{\mathbb{V}}^{(\mathbb{B})} : \llbracket y = x\uparrow \rrbracket = \mathbb{1}\}$ is the *ascent* of x . For the corresponding element in the separated universe $\mathbb{V}^{(\mathbb{B})}$ the same name and notation are preserved.

1.2.7. Let $X, Y, f \in \mathcal{P}(\mathbb{V}^{(\mathbb{B})})$ and f be a mapping from X to Y . There exists a mapping $f\uparrow$ from $X\uparrow$ to $Y\uparrow$ within $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$\llbracket f\uparrow(x) = f(x) \rrbracket = \mathbb{1} \quad (x \in X),$$

if and only if f is *extensional*, i. e., the relation holds:

$$\llbracket x = x' \rrbracket \leq \llbracket f(x) = f(x') \rrbracket \quad (x, x' \in X).$$

The map $f\uparrow$ with the above property is unique and satisfy the relation $f\uparrow(A\uparrow) = f(A)\uparrow$ ($A \subset X$). The composite of extensional maps is extensional. Moreover, the ascent of a composite is equal to the composite of the ascents inside $\mathbb{V}^{(\mathbb{B})}$:

$$\mathbb{V}^{(\mathbb{B})} \models (g \circ f)\uparrow = g\uparrow \circ f\uparrow.$$

Observe also that if f and f^{-1} are extensional then $(f\uparrow)^{-1} = (f^{-1})\uparrow$.

1.2.8. Suppose that $X \in \mathbb{V}$, $X \neq \emptyset$; i. e., X is a nonempty set. Let the letter ι denote the standard name embedding $x \mapsto x^\wedge$ ($x \in X$). Then $\iota(X)\uparrow = X^\wedge$ and $X = \iota^{-1}(X^\wedge\downarrow)$. Take $Y \in \mathbb{V}^{(\mathbb{B})}$ with $\llbracket Y \neq \emptyset \rrbracket = \mathbb{1}$. Using the above relations, we may extend the ascent operation to the case of maps from X to $Y\downarrow$ and descent operation to the case of internal maps from X^\wedge to Y with $\llbracket Y \neq \emptyset \rrbracket = \mathbb{1}$.

The maps $f\uparrow$ and $g\downarrow$ are called *modified descent* of f and *modified ascent* of g , respectively. (Again, when there is no ambiguity, we simply speak of ascents and use simple arrows.) It is easy to see that $g\uparrow$ is the unique map from X to $Y\downarrow$ satisfying

$$g\downarrow(x) = g(x^\wedge)\downarrow \quad (x \in X)$$

and $f\uparrow$ is the unique map from X^\wedge to Y within $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$\llbracket f\uparrow(x^\wedge) = f(x) \rrbracket = \mathbb{1} \quad (x \in X).$$

1.2.9. Given $X \subset \mathbb{V}^{(\mathbb{B})}$, we denote by $\text{mix}(X)$ the set of all mixtures of the form $\text{mix}(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following assertions are referred to as the *rules for canceling arrows* or the *Escher rules*.

Let X and X' be subsets of $\mathbb{V}^{(\mathbb{B})}$ and let $f : X \rightarrow X'$ be an extensional mapping. Suppose that $Y, Y', g \in \mathbb{V}^{(\mathbb{B})}$ are such that $\llbracket Y \neq \emptyset \rrbracket = \llbracket g : Y \rightarrow Y' \rrbracket = \mathbb{1}$. Then

$$\begin{aligned} X\uparrow\downarrow &= \text{mix}(X), & Y\downarrow\uparrow &= Y; \\ f\uparrow\downarrow &= f, & g\downarrow\uparrow &= g. \end{aligned}$$

There are some other cancelation rules.

1.3. Boolean Valued Reals and Vector Lattices

The main results of the section tells us that the Boolean valued interpretation of the field of reals (complexes) is a real (complex) universally complete vector lattice. Everywhere below \mathbb{B} is a complete Boolean algebra and $\mathbb{V}^{(\mathbb{B})}$ is the corresponding Boolean valued universe.

1.3.1. By virtue of the Transfer and Maximum Principles there exists an element $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ for which $\llbracket \mathcal{R} \text{ is a field of reals} \rrbracket = 1$. Note also that $\varphi(x)$, formally presenting the expressions of the axioms of an archimedean ordered field x , is bounded; therefore, by the Restricted Transfer Principle $\llbracket \varphi(\mathbb{R}^\wedge) \rrbracket = 1$, i. e., $\llbracket \mathbb{R}^\wedge \text{ is an archimedean ordered field} \rrbracket = 1$. Thus, we will assume that \mathbb{R}^\wedge is a dense subfield of \mathcal{R} , while the elements $0 := 0^\wedge$ and $1 := 1^\wedge$ are the zero and unity of \mathcal{R} within the model $\mathbb{V}^{(\mathbb{B})}$.

1.3.2. Let \mathbb{R} be the underlying set of the field \mathcal{R} on which the addition \oplus , multiplication \otimes , and ordering \leq are given. Then \mathcal{R} is a 5-tuple $(\mathbb{R}, \oplus, \otimes, \leq, 0^\wedge, 1^\wedge)$ within $\mathbb{V}^{(\mathbb{B})}$; in symbols, $\mathbb{V}^{(\mathbb{B})} \models \mathcal{R} = (\mathbb{R}, \oplus, \otimes, \leq, 0^\wedge, 1^\wedge)$.

The descent $\mathcal{R}\downarrow$ of the field \mathcal{R} is the descent of the underlying set $\mathbb{R}\downarrow$ together with the descended operations $+: = \oplus\downarrow$, $\cdot := \otimes\downarrow$, order relation $\leq := \leq\downarrow$, and distinguished elements $0^\wedge, 1^\wedge$; in symbols, $\mathcal{R}\downarrow = (\mathbb{R}\downarrow, \oplus\downarrow, \otimes\downarrow, \leq\downarrow, 0^\wedge, 1^\wedge)$. Also, we may introduce multiplication by the standard reals in $\mathcal{R}\downarrow$ by the rule

$$y = \lambda x \iff \llbracket y = \lambda^\wedge \odot x \rrbracket = 1 \quad (\lambda \in \mathbb{R}, x, y \in \mathcal{R}\downarrow).$$

1.3.3. Gordon Theorem. *Let \mathcal{R} be the reals in $\mathbb{V}^{(\mathbb{B})}$. Then $\mathcal{R}\downarrow$ (with the descended operations and order) is a universally complete vector lattice with a weak order unit $1 := 1^\wedge$. Moreover, there exists a Boolean isomorphism χ of \mathbb{B} onto the base $\mathbb{P}(\mathcal{R}\downarrow)$ such that for all $x, y \in \mathcal{R}\downarrow$ and $b \in \mathbb{B}$ we have*

$$\begin{aligned} \chi(b)x &= \chi(b)y \iff b \leq \llbracket x = y \rrbracket, \\ \chi(b)x &\leq \chi(b)y \iff b \leq \llbracket x \leq y \rrbracket. \end{aligned} \tag{G}$$

1.3.4. A vector lattice is an *f-algebra* if it is simultaneously a real algebra and satisfies, for all $a, x, y \in X_+$, the conditions: 1) $x \geq 0$ and $y \geq 0$ imply $xy \geq 0$ and 2) $x \perp y = 0$ implies that $(ax) \perp y$ and $(xa) \perp y$. The multiplication in every (Archimedean) *f-algebra* is commutative and associative. An *f-algebra* is called *semi-prime* if $xy = 0$ implies $x \perp y$ for all x and y . The universally complete vector lattice $\mathcal{R}\downarrow$ with the descended multiplications is a semiprime *f-algebra* with a ring unit $1 = 1^\wedge$.

1.3.5. By the Maximum Principle 1.1.7, there is an element $\mathcal{C} \in \mathbb{V}^{(\mathbb{B})}$ for which $\llbracket \mathcal{C} \text{ is the complexes} \rrbracket = 1$. Since the equality $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ is expressed by a bounded set-theoretic formula, from the Restricted Transfer Principle 1.2.2 we obtain $\llbracket \mathbb{C}^\wedge = \mathbb{R}^\wedge \oplus i^\wedge \mathbb{R}^\wedge \rrbracket = 1$. Moreover, \mathbb{R}^\wedge is assumed to be a dense subfield of \mathcal{R} ; therefore, we can also assume that \mathbb{C}^\wedge is a dense subfield of \mathcal{C} . If 1 is the unity of \mathbb{C} then 1^\wedge is the unity of \mathcal{C} inside $\mathbb{V}^{(\mathbb{B})}$. We write i instead of i^\wedge and 1 instead of 1^\wedge . By the Gordon Theorem $\mathcal{C}\downarrow = \mathcal{R}\downarrow \oplus i\mathcal{R}\downarrow$; consequently $\mathcal{C}\downarrow$ is a universally complete complex vector lattice, i. e., the complexification of a vector lattice $\mathcal{R}\downarrow$. Moreover, $\mathcal{C}\downarrow$ is a complex *f-algebra* defined as the complexification of a real *f-algebra* with a ring unit $1 := 1^\wedge$.

1.3.6. Let A be an f -algebra. A vector lattice X is said to be an f -module over A if the following hold:

- (1) X is a module over A (with respect to a multiplication $A \times X \ni (a, x) \mapsto ax \in X$);
- (2) $ax \geq 0$ for all $a \in A_+$ and $x \in X_+$;
- (3) $x \perp y$ implies $ax \perp y$ for all $a \in A_+$ and $x, y \in X$.

A vector lattice X has a natural f -module structure over $\text{Orth}(X)$, i. e., $\pi x := \pi(x)$ for all $x \in X$ and $\pi \in \text{Orth}(X)$. Clearly, X is an f -module over an arbitrary f -submodule $A \subset \text{Orth}(X)$ and, in particular, over $\mathcal{Z}(X)$.

1.3.7. Theorem *Let X be an f -module over $\mathcal{Z}(Y)$ with Y being a Dedekind complete vector lattice and $\mathbb{B} = \mathbb{P}(Y)$. Then there exists $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ such that $\llbracket \mathcal{X} \text{ is a vector lattice over } \mathcal{Z} \rrbracket = 1$, $\mathcal{X} \downarrow$ is an f -module over A^u , and there is an f -module isomorphism h from X to $\mathcal{X} \downarrow$ satisfying $\mathcal{X} \downarrow = \text{mix}(h(X))$.*

1.3.8. Theorem 1.3.3 was established in [22]. The concept of an f -module was introduced in [62].

1.4. Boolean Valued Functionals

We will demonstrate in this section how Boolean valued analysis works by transferring some results from order bounded functionals to operators. Below X and Y stand for vector lattices, where Y is an order dense sublattice in $\mathcal{Z} \downarrow$.

1.4.1. Let \mathbb{B} be a complete Boolean algebra and \mathcal{Z} be the field of reals in $\mathbb{V}^{(\mathbb{B})}$. The fact that X is a vector lattice over the ordered field \mathbb{R} may be rewritten as a restricted formula, say, $\varphi(X, \mathbb{R})$. Hence, recalling the Restricted Transfer Principle 1.2.2, we come to the identity $\llbracket \varphi(X^\wedge, \mathbb{R}^\wedge) \rrbracket = 1$ which amounts to saying that X^\wedge is a vector lattice over the ordered field \mathbb{R}^\wedge inside $\mathbb{V}^{(\mathbb{B})}$.

Let $X^{\wedge\sim} := L^\sim(X^\wedge, \mathcal{Z})$ be the space of order bounded \mathbb{R}^\wedge -linear functionals from X^\wedge to \mathcal{Z} . More precisely, \mathcal{Z} is considered as a vector space over the field \mathbb{R}^\wedge and by Maximum Principle there exists $X^{\wedge\sim} \in \mathbb{V}^{(\mathbb{B})}$ such that $\llbracket X^{\wedge\sim} \text{ is a vector space over } \mathcal{Z} \text{ of } \mathbb{R}^\wedge\text{-linear order bounded functionals from } X^\wedge \text{ to } \mathcal{Z} \rrbracket = 1$. A functional $\tau \in X^{\wedge\sim}$ is positive if $\llbracket (\forall x \in X^\wedge) \tau(x) \geq 0 \rrbracket = 1$.

1.4.2. It can be easily seen that the Riesz–Kantorovich Theorem remains true if X is a vector lattice over a dense subfield $\mathbb{P} \subset \mathbb{R}$ and Y be a Dedekind complete vector lattices (over \mathbb{R}) and $L^\sim(X, Y)$ is replaced by $L_{\mathbb{P}}^\sim(X, Y)$, the vector spaces over \mathbb{R} of all \mathbb{P} -linear order bounded operators from X to Y , ordered by the cone of positive operators: $L_{\mathbb{P}}^\sim(X, Y)$ is a Dedekind complete vector lattice. Thus, the descent $X^{\wedge\sim} \downarrow$ of $X^{\wedge\sim}$ is a Dedekind complete vector lattice. Boolean valued interpretation of this fact yields that $X^\wedge := L_{\mathbb{R}^\wedge}^\sim(X^\wedge, \mathbb{R})$ is a Dedekind complete vector lattice within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} := \mathbb{P}(Y)$. In particular, the descent $X^{\wedge\sim} \downarrow$ of the space $X^{\wedge\sim}$ is Dedekind complete vector lattice. Let $L_{dp}^\sim(X, Y)$ and $\text{Hom}(X, Y)$ stand respectively for the space of disjointness preserving order bounded operators and the set of all lattice homomorphism from X to Y . The following result is based on the construction from 1.2.8.

1.4.3. Theorem. Let X and Y be vector lattices with Y universally complete and represented as $Y = \mathcal{R}\downarrow$. Given $T \in L^\sim(X, Y)$, the modified ascent $T\uparrow$ is an order bounded \mathbb{R}^\wedge -linear functional on X^\wedge within $V^{(\mathbb{B})}$; i. e., $\llbracket T\uparrow \in X^{\wedge\sim} \rrbracket = 1$. The mapping $T \mapsto T\uparrow$ is a lattice isomorphism between the Dedekind complete vector lattices $L^\sim(X, Y)$ and $X^{\wedge\sim}\downarrow$.

1.4.4. Corollary. Given operators $R, S \in L^\sim(X, Y)$, put $\sigma := S\uparrow$ and $\tau := T\uparrow$. The following are hold true:

- (1) $S \leq T \iff \llbracket \sigma \leq \tau \rrbracket = 1$;
- (2) $S = |T| \iff \llbracket \sigma = |\tau| \rrbracket = 1$;
- (3) $S \perp T \iff \llbracket \sigma \perp \tau \rrbracket = 1$;
- (4) $\llbracket T \in \text{Hom}(X, Y) \rrbracket \iff \llbracket \tau \in \text{Hom}(X^\wedge, \mathcal{R}) \rrbracket = 1$;
- (5) $T \in L_{dp}^\sim(X, Y) \iff \llbracket \tau \in (X^{\wedge\sim})_{dp} \rrbracket = 1$.

1.4.5. Consider a vector lattice X and let D be an order ideal in X . A linear operator T from D into X is *band preserving* provided that one (and hence all) of the following holds: $x \perp y$ implies $Tx \perp y$ for all $x \in D$ and $y \in X$, or equivalently, $Tx \in \{x\}^{\perp\perp}$ for all $x \in D$ (the disjoint complements are taken in X). If X is a vector lattice with the principal projection property and $D \subset X$ is an order dense ideal, then a linear operator $T : D \rightarrow X$ is band preserving if and only if T commutes with band projections: $\pi Tx = T\pi x$ for all $\pi \in \mathbb{P}(X)$ and $x \in D$.

1.4.6. Let $\text{End}_N(X_{\mathbb{C}})$ be the set of all band preserving endomorphisms of $X_{\mathbb{C}}$ with $X := \mathcal{R}\downarrow$. Clearly, $\text{End}_N(XG_{\mathbb{C}})$ is a complex vector space. Moreover, $\text{End}_N(X_{\mathbb{C}})$ becomes a faithful unitary module over the ring $X_{\mathbb{C}}$ on letting gT be equal to $gT : x \mapsto g \cdot Tx$ for all $x \in X_{\mathbb{C}}$. This is immediate since the multiplication by an element of $X_{\mathbb{C}}$ is band preserving and the composite of band preserving operators is band preserving too.

1.4.7. By $\text{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ we denote the element of $V^{(\mathbb{B})}$ that represents the space of all \mathbb{C}^\wedge -linear operators from \mathcal{C} into \mathcal{C} . Then $\text{End}_{\mathbb{C}^\wedge}(\mathcal{C})$ is a vector space over \mathcal{C} inside $V^{(\mathbb{B})}$, and $\text{End}_{\mathbb{C}^\wedge}(\mathcal{C})\downarrow$ is a faithful unitary module over a complex f -slgebra $X_{\mathbb{C}}$.

1.4.8. A linear operator T on a universally complete vector lattice X or $X_{\mathbb{C}}$ is *band preserving* if and only if T is *extensional*.

\triangleleft Take a linear operator $T : X \rightarrow X$. By the Gordon Theorem the extensionality condition $\llbracket x = y \rrbracket \leq \llbracket Tx = Ty \rrbracket$ ($x, y \in X = \mathcal{R}\downarrow$) amounts to saying that the identity $\pi x = \pi y$ implies $\pi Tx = \pi Ty$ for all $x, y \in X$ and $\pi \in \mathbb{P}(X)$. By linearity of T the latter is equivalent to $\pi x = 0 \implies \pi Tx = 0$ ($x \in X$, $\pi \in \mathbb{P}(X)$). Substituting $y := \pi^\perp x$ yields $\pi T \pi^\perp = 0$ or, which is the same, $\pi T = \pi T \pi$. According to 1.4.5 T is band preserving. The complex case is treated by complexification. \triangleright

1.4.9. Theorem. The modules $\text{End}_N(X_{\mathbb{C}})$ and $\text{End}_{\mathbb{C}^\wedge}(\mathcal{C})\downarrow$ are isomorphic. The isomorphism may be established by sending a band preserving operator to its ascent. The same remains true when \mathcal{C} and \mathbb{C} are replaced by \mathcal{R} and \mathbb{R} , respectively.

\triangleleft By virtue of 1.4.8, we can apply the constructions 1.2.4 and 1.2.7, as well

as the cancelation rules 1.2.9. \triangleright

Chapter 2. Band Preserving Operators

2.1. Wickstead's Problem and Cauchy's Functional Equation

In this section we demonstrate that the band preserving operators in universally complete vector lattices are solutions in disguise of the Cauchy functional equation and the Wickstead problem amounts to that of regularity of all solutions to the equation.

2.1.1. The Wickstead Problem: *When are we so happy in a vector lattice that all band preserving linear operators turn out to be order bounded?*

This question was raised by Wickstead in [83]. Further progress is presented in [1, 2, 29, 43, 44, 67]. Combined approach based on logical, algebraic, and analytic tools was presented in [43, 44, 45]. A survey of the main ideas and results on the problem and its modifications see in [31].

The answer depends on the vector lattice in which the operator in question acts. Therefore, the problem can be reformulated as follows: *Characterize the vector lattices in which every band preserving linear operators is order bounded.*

Let X be a universally complete vector lattice and T a band preserving linear operator in X . By the Gordon Theorem we may assume that $X = \mathcal{R}\downarrow$, where \mathcal{R} is the field of reals within $\mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} = \mathbb{P}(X)$. Moreover, according to Theorem 1.4.9 we can assume further that $T = \tau\downarrow$, where $\tau \in \mathbb{V}^{(\mathbb{B})}$ is an internal \mathbb{R}^\wedge -linear function from \mathcal{R} to \mathcal{R} . It can be easily seen that T is order bounded if and only if $\llbracket \tau \text{ is order bounded (i.e., } \tau \text{ is bounded on intervals } [a, b] \subset \mathcal{R} \rrbracket = 1$.

2.1.2. By \mathbb{F} we denote either \mathbb{R} or \mathbb{C} . The *Cauchy functional equation* with $f : \mathbb{F} \rightarrow \mathbb{F}$ unknown has the form

$$f(x + y) = f(x) + f(y) \quad (x, y \in \mathbb{F}).$$

It is easy that a solution to the equation is automatically \mathbb{Q} -homogeneous, i.e. it satisfies another functional equation:

$$f(qx) = qf(x) \quad (q \in \mathbb{Q}, x \in \mathbb{F}).$$

In the sequel we will be interested in a more general situation. Namely, we will consider the simultaneous functional equations

$$\begin{cases} f(x + y) = f(x) + f(y) & (x, y \in \mathbb{F}), \\ f(px) = pf(x) & (p \in \mathbb{P}, x \in \mathbb{F}), \end{cases} \quad (L)$$

where \mathbb{P} is a subfield of \mathbb{F} which includes \mathbb{Q} . Denote by $\mathbb{F}_{\mathbb{P}}$ the field \mathbb{F} which is considered as a vector space over \mathbb{P} . Clearly, solutions to the simultaneous equations (L) are precisely \mathbb{P} -linear functions from $\mathbb{F}_{\mathbb{P}}$ to $\mathbb{F}_{\mathbb{P}}$.

2.1.3. Let \mathcal{E} be a Hamel basis for a vector space $\mathbb{F}_{\mathbb{P}}$, and let $\mathcal{F}(\mathcal{E}, \mathbb{F})$ be the space of all functions from \mathcal{E} to \mathbb{F} . The solution set of (L) is a vector space

over \mathbb{F} isomorphic with $\mathcal{F}(\mathcal{E}, \mathbb{F})$. Such an isomorphism can be implemented by sending a solution f to the restriction $f|_{\mathcal{E}}$ of f to \mathcal{E} . The inverse isomorphism $\varphi \mapsto f_{\varphi}$ ($\varphi \in \mathcal{F}(\mathcal{E}, \mathbb{F})$) is defined by

$$f_{\varphi}(x) := \sum_{e \in \mathcal{E}} \varphi(e) \psi(e) \quad (x \in \mathbb{F}_{\mathbb{P}}),$$

where $x = \sum_{e \in \mathcal{E}} \psi(e)e$ is the expansion of x with respect to Hamel basis \mathcal{E} .

2.1.4. Theorem *Each solution of (L) is either \mathbb{F} -linear or everywhere dense in $\mathbb{F}^2 := \mathbb{F} \times \mathbb{F}$. In particular if \mathcal{E} is a Hamel basis for a vector space $\mathbb{F}_{\mathbb{P}}$ and f be the unique extension of a function $\varphi : \mathcal{E} \rightarrow \mathbb{R}$. Then f is continuous if and only if $\varphi(e)/e = \text{const}$ ($e \in \mathcal{E}$).*

2.1.5. Assume now that $\mathbb{F} = \mathbb{C}$ and $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ with \mathbb{P}_0 being a subfield in \mathbb{R} . Then the space of solutions of the system (L) is a complexification of the space of solution of the same system with $\mathbb{P} := \mathbb{P}_0$. In more detail, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is a \mathbb{P}_0 -linear function then we have the unique \mathbb{P} -linear function $\tilde{g} : \mathbb{C} \rightarrow \mathbb{C}$ defined as

$$\tilde{g}(z) = g(x) + ig(y) \quad (z = x + iy \in \mathbb{C}).$$

Conversely, if $f : \mathbb{C} \rightarrow \mathbb{C}$ is a \mathbb{P} -linear function then we have the a unique pair of \mathbb{P}_0 -linear functions $g_1, g_2 : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(z) = \tilde{g}_1(z) + i\tilde{g}_2(z)$ ($z \in \mathbb{C}$). Thus, every solution f of (L) can be represented in the form $f = f_1 + if_2$, where $f_1, f_2 : \mathbb{C} \rightarrow \mathbb{C}$ are \mathbb{P}_0 -linear and $f_i(\mathbb{R}) \subset \mathbb{R}$ ($i = 1, 2$) have the same property. Say that f is *monotone* or *bounded* if f_1 and f_2 .

2.1.6. Let \mathbb{P} be a subfield of \mathbb{F} , while $\mathbb{P} := \mathbb{P}_0 + i\mathbb{P}_0$ for some dense subfield $\mathbb{P}_0 \subset \mathbb{R}$, in case $\mathbb{F} = \mathbb{C}$. The following are equivalent:

- (1) $\mathbb{F} = \mathbb{P}$;
- (2) every solution to (L) is order bounded.

◁ The implication (1) \implies (2) is trivial. Prove the converse by way of contradiction. The assumption that $\mathbb{F} \neq \mathbb{P}$ implies that each Hamel basis \mathcal{E} for the vector space $\mathbb{F}_{\mathbb{P}}$ contains at least two nonzero distinct elements $e_1, e_2 \in \mathcal{E}$. Define the function $\psi : \mathcal{E} \rightarrow \mathbb{F}$ so that $\psi(e_1)/e_1 \neq \psi(e_2)/e_2$. Then the \mathbb{P} -linear function $f = f_{\psi} : \mathbb{F} \rightarrow \mathbb{F}$, coinciding with ψ on \mathcal{E} , would exist by 2.1.2 and be discontinuous by 2.1.4. ▷

2.1.7. Add to the system (L) the equation $f(xy) = f(x)f(y)$ (or $f(xy) = f(x)y + xf(y)$) ($x, y \in \mathbb{F}$). A solution of the resulting system is called \mathbb{P} -*endomorphism* (\mathbb{P} -*derivation*). The existence of the nontrivial \mathbb{P} -endomorphism and \mathbb{P} -derivation can be obtained similarly, but using a transcendental basis instead of a Hamel basis (cp. [37]). Interpreting such existence results in a Boolean valued model yields the existence of band preserving endomorphism and derivations of a universally complete f -algebra, see [44, 45], as well as [49].

2.2. Locally One-Dimensional Vector Lattices

Boolean valued representation of a vector lattice is a vector sublattice in \mathcal{B} considered as a vector lattice over \mathbb{R}^{\wedge} . It stands to reason to find out what

construction in a vector lattice corresponds to a Hamel basis for its Boolean valued representation.

2.2.1. Let X be a vector lattice with a cofinal family of band projections. We will say that $x, y \in X$ *differ* at $\pi \in \mathbb{P}(X)$ provided that $\pi|x - y|$ is a weak order unit in $\pi(X)$ or, equivalently, if $\pi(X) \subset |x - y|^{\perp\perp}$. Clearly, x and y differ at π whenever $\rho x = \rho y$ implies $\pi\rho = 0$ for all $\rho \in \mathbb{P}(X)$. A subset \mathcal{E} of X is said to be *locally linearly independent* provided that, for an arbitrary nonzero band projection π in X and each collection of the elements $e_1, \dots, e_n \in \mathcal{E}$ that are pairwise different at π , and each collection of reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, the condition $\pi(\lambda_1 e_1 + \dots + \lambda_n e_n) = 0$ implies that $\lambda_k = 0$ for all $k := 1, \dots, n$. In other words, \mathcal{E} is locally linearly independent if for every band projection $\pi \in \mathbb{P}(X)$ any subset of $\pi(\mathcal{E})$ consisting of nonzero members pairwise different at π is linearly independent.

An inclusion-maximal locally linearly independent subset of X is called a *local Hamel basis* for X .

2.2.2. Each vector lattice X with a cofinal family of band projections has a local Hamel basis for X .

2.2.3. A locally linearly independent set \mathcal{E} in G is a local Hamel basis if and only if for every $x \in G$ there exist a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathbb{P}(G)$ and a family of reals $(\lambda_{\xi,e})_{\xi \in \Xi, e \in \mathcal{E}}$ such that

$$x = o\text{-}\sum_{\xi \in \Xi} \left(\sum_{e \in \mathcal{E}} \lambda_{\xi,e} \pi_\xi e \right)$$

and for every $\xi \in \Xi$ the set $\{e \in \mathcal{E} : \lambda_{\xi,e} \neq 0\}$ is finite and consists of nonzero elements pairwise different at π_ξ . Moreover, the representation is unique up to refinements of the partition of unity; cp. [4, §6] and [41, §5.1]. The following result of [39, Proposition 4.6 (1)] explains why and how the concept of local Hamel basis is a so useful technical tool (cp. [4]).

2.2.4. Assume that $\mathcal{E}, \mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$, $\llbracket \mathcal{E} \subset \mathcal{X} \rrbracket = 1$, $\llbracket \mathcal{X} \text{ is a vector subspace of } \mathcal{R}_{\mathbb{R}} \rrbracket = 1$, and $X := \mathcal{X} \downarrow$. Then $\llbracket \mathcal{E} \text{ is a Hamel basis for the vector space } \mathcal{X} \text{ (over } \mathbb{R}^\wedge) \rrbracket = 1$ if and only if $\mathcal{E} \downarrow$ is a local Hamel basis for X .

2.2.5. A vector lattice X is said to be *locally one-dimensional* if for every two nondisjoint $x_1, x_2 \in X$ there exist nonzero components u_1 and u_2 of x_1 and x_2 respectively such that u_1 and u_2 are proportional; cp. [4, Definition 11.1]. Equivalent definitions see in [41, Proposition 5.1.2].

2.2.6. Let X be a laterally complete vector lattice and $\mathcal{X} \in \mathbb{V}^{(\mathbb{B})}$ be its Boolean valued representation with $\mathbb{B} := \mathbb{P}(X)$. Then X is locally one-dimensional if and only if \mathcal{X} is one-dimensional vector lattice over \mathbb{R}^\wedge within $\mathbb{V}^{(\mathbb{B})}$, i. e., $\llbracket \mathcal{X} = \mathbb{R}^\wedge \rrbracket = 1$.

2.2.7. A universally complete vector lattice is locally one-dimensional if and only if every band preserving linear operator in it is order bounded.

◁ By the Gordon Theorem we can assume that $X = \mathcal{R} \downarrow$ with $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ and $\mathbb{B} \simeq \mathbb{P}(X)$. Thus, the problem reduces to existence of a discontinuous solution to the Cauchy functional equation (L) and the claim follows from 2.1.6. ▷

2.2.8. Let \mathbb{R} is a transcendental extension of a subfield $\mathbb{P} \subset \mathbb{R}$. There exists an \mathbb{P} -linear subspace \mathcal{X} in \mathbb{R} such that \mathcal{X} and \mathbb{R} are isomorphic vector spaces over \mathbb{P} but they are not isomorphic as ordered vector spaces over \mathbb{P} .

◁ Let \mathcal{E} be a Hamel basis of a \mathbb{P} -vector space \mathbb{R} . Since \mathcal{E} is infinite, we can choose a proper subset $\mathcal{E}_0 \subsetneq \mathcal{E}$ of the same cardinality: $|\mathcal{E}_0| = |\mathcal{E}|$. If \mathcal{X} denotes the \mathbb{P} -subspace of \mathbb{R} generated by \mathcal{E}_0 , then $\mathcal{X}_0 \subsetneq \mathbb{R}$ and \mathcal{X} and \mathbb{R} are isomorphic as vector spaces over \mathbb{P} . If \mathcal{X} and \mathbb{R} were isomorphic as ordered vector spaces over \mathbb{P} , then \mathcal{X} would be order complete and, in consequence, we would have $\mathcal{X} = \mathbb{R}$, a contradiction. ▷

2.2.9. Theorem. Let X be a nonlocally one-dimensional universally complete vector lattice. Then there exist a vector sublattice $X_0 \subset X$ and a band preserving linear bijection $T : X_0 \rightarrow X$ such that T^{-1} is also band preserving but X_0 and X are not lattice isomorphic.

◁ We can assume without loss of generality that $X = \mathcal{R}\downarrow$ and $[\mathcal{R} \neq \mathbb{R}^\wedge] = 1$. By 4.6.5 there exist an \mathbb{R}^\wedge -linear subspace \mathcal{X} in \mathcal{R} and \mathbb{R}^\wedge -linear isomorphism τ from \mathcal{X} onto \mathcal{R} , while \mathcal{X} and \mathcal{R} are not isomorphic as ordered vector spaces over \mathbb{R}^\wedge . Put $X_0 := \mathcal{X}\downarrow$, $T := \tau\downarrow$ and $S := \tau^{-1}\downarrow$. The maps S and T are band preserving and linear. Moreover, $S = (\tau\downarrow)^{-1} = T^{-1}$. It remains to observe that X_0 and X are lattice isomorphic if and only if \mathcal{X} and \mathcal{R} are isomorphic as ordered vector spaces. ▷

2.2.10. Let γ be a cardinal. A vector lattice X is said to be *Hamel γ -homogeneous* whenever there exists a local Hamel basis of cardinality γ in X consisting of strongly distinct weak order units. Two elements $x, y \in X$ are said to be *strongly distinct* if $|x - y|$ is a weak order unit in X .

2.2.11. Let X be a universally complete vector lattice. There is a band X_0 in X such that X_0^\perp is locally one-dimensional and there exists a partition of unity $(\pi_\gamma)_{\gamma \in \Gamma}$ in $\mathbb{P}(X_0)$ with Γ a set of infinite cardinals such that $\pi_\gamma X_0$ is Hamel γ -homogeneous for all $\gamma \in \Gamma$.

2.2.12. A local Hamel basis is also called a *d-basis*. This concept stems from [17], but for the first time in the context of disjointness preserving operators was used in [1, 2]. Various aspects of the concept can be found in [4, 5]. Theorem 2.2.6 was established in [29], while 2.2.7 in [2, 67]. Another proof of Theorem 2.2.8 one can find in [5].

2.3. Algebraic Band Preserving Operators

In this section some description of algebraic orthomorphisms on a vector lattice is given and the Wickstead problem for algebraic operators is examined.

2.3.1. Let $\mathbb{P}[x]$ be a ring of polynomials in variable x over a field \mathbb{P} . An operator T on a vector space X over a field \mathbb{P} is said to be *algebraic* if there exists a nonzero $\varphi \in \mathbb{P}[x]$, a polynomials with coefficients in \mathbb{P} , for which $\varphi(T) = 0$.

For an algebraic operator T there exists a unique polynomial φ_T such that $\varphi_T(T) = 0$, the leading coefficient of φ_T equals to 1, and φ_T divides each polynomial ψ with $\psi(T) = 0$. The polynomial φ_T is called the *minimal polynomial*

of T . The simple examples of algebraic operators yield a projection P (an idempotent operator, $P^2 = P$) in X with $\varphi_P(\lambda) = \lambda^2 - \lambda$ whenever $P \neq 0, I_X$, and a nilpotent operator S ($S^m = 0$ for some $m \in \mathbb{N}$) in X with $\varphi_S(\lambda) = \lambda^k$, $k \leq m$.

For an operator T on X , the set of all eigenvalues of T will be denoted throughout by $\sigma_p(T)$. A real number λ is a root of φ_T if and only if $\lambda \in \sigma_p(T)$. In particular, $\sigma_p(T)$ is finite. If $b - a^2 > 0$ for some $a, b \in \mathbb{R}$ then $T^2 + 2aT + bI$ is a weak order unit in $\text{Orth}(X)$ for every $T \in \text{Orth}(X)$; cp. [14].

2.3.2. *Let X be a vector lattice and T in $\text{Orth}(X)$ is algebraic. Then*

$$\varphi_T(x) = \prod_{\lambda \in \sigma_p(T)} (x - \lambda).$$

2.3.3. *Consider the universally complete vector lattice $X = \mathcal{R}\downarrow$. Let T be a band preserving linear operator on X and τ an \mathbb{R}^\wedge -linear function on \mathcal{R} . For $\varphi \in \mathbb{R}[x]$, $\varphi(x) = a_0 + a_1x + \dots + a_nx^n$ define $\hat{\varphi} \in \mathbb{R}^\wedge[x]$ by $\hat{\varphi}(x) = a_0^\wedge + a_1^\wedge x + \dots + a_n^\wedge x^{n^\wedge}$. Then*

$$\hat{\varphi}(\tau)\downarrow = \varphi(\tau\downarrow), \quad \varphi(T)\uparrow = \hat{\varphi}(T\uparrow).$$

\triangleleft It follows from 1.2.4 and 1.2.7 that $(\tau^{n^\wedge})\downarrow = (\tau\downarrow)^n$ and $(T^n)\uparrow = (T\uparrow)^{n^\wedge}$. Thus, it remains to apply 1.4.9. \triangleright

2.3.4. A linear operator T on a vector lattice X is said to be *diagonal* if $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ for some collections of reals $\lambda_1, \dots, \lambda_m$ and projection operators P_1, \dots, P_m on X with $P_i \circ P_j = 0$ ($i \neq j$). In the equality above, we may and will assume that $P_1 + \dots + P_m = I_X$ and that $\lambda_1, \dots, \lambda_m$ are pairwise different. An algebraic operator T is diagonal if and only if the minimal polynomial of T have the form $\varphi_T(x) = (x - \lambda_1) \cdot \dots \cdot (x - \lambda_m)$ with pairwise different $\lambda_1, \dots, \lambda_m \in \mathbb{R}$.

We call an operator T on X *strongly diagonal* if there exist pairwise disjoint band projections P_1, \dots, P_m and real numbers $\lambda_1, \dots, \lambda_m$ such that $T = \lambda_1 P_1 + \dots + \lambda_m P_m$. In particular, every strongly diagonal operator on X is an orthomorphism.

2.3.5. *Let $T = \lambda_1 P_1 + \dots + \lambda_m P_m$ be a diagonal operator on a vector lattice X . Then T is band preserving if and only if the projection operators P_1, \dots, P_m are band preserving.*

\triangleleft The sufficiency is obvious. To prove the necessity, observe first that if T is band preserving then so is T^n for all $n \in \mathbb{N}$ and thus $\varphi(T)$ is band preserving for every polynomial $\varphi \in \mathbb{R}[x]$. Next, make use of the representation $P_j = \varphi_j(T)$ ($j := 1, \dots, m$), where $\varphi_j \in \mathbb{R}[x]$ is an interpolation polynomial defined by $\varphi_j(\lambda_k) = \delta_{jk}$ with δ_{jk} the Kronecker symbol. \triangleright

2.3.6. Theorem. *Let X be a universally complete vector lattice. The following are equivalent:*

- (1) *The Boolean algebra $\mathbb{P}(X)$ is σ -distributive.*
- (2) *Every algebraic band preserving operator in X is order bounded.*
- (3) *Every algebraic band preserving operator in X is strongly diagonal.*
- (4) *Every band preserving diagonal operator in X is strongly diagonal.*

- (5) Every band preserving nilpotent operator in X is order bounded.
 (6) Every band preserving nilpotent operator in X is trivial.

◁ The only nontrivial implications are (2) \implies (3) and (6) \implies (2).

(2) \implies (3) We have to prove that an algebraic orthomorphism on X is strongly diagonal. Let T be an orthomorphism in X and $\varphi(T) = 0$, where φ is a minimal polynomial of T , so that $\varphi(\lambda) = (\lambda - \lambda_1) \cdot \dots \cdot (\lambda - \lambda_m)$ with $\lambda_1, \dots, \lambda_m \in \mathbb{R}$. Since T admits a unique extension to an orthomorphism on X^u , we can assume without loss of generality that $X = X^u = \mathcal{R}\downarrow$ and $\tau = T\uparrow$. Then $\llbracket \tau(x) = \lambda_0 x \ (x \in \mathcal{R}) \rrbracket = \mathbb{1}$ for some $\lambda_0 \in \mathcal{R}$. It is seen from 2.3.3 that $\hat{\varphi}(\lambda_0) = 0$ and thus $(\lambda_0 - \lambda_1^\wedge) \cdot \dots \cdot (\lambda_0 - \lambda_m^\wedge) = 0$ or $\lambda_0 \in \{\lambda_1^\wedge, \dots, \lambda_m^\wedge\}$ within $\mathbb{V}^{(\mathbb{B})}$. Put $P_l := \chi(b_l)$ with $b_l := \llbracket \lambda_0 = \lambda_l^\wedge \rrbracket$ and observe that $\{P_1, \dots, P_m\}$ is a partition of unity in $\mathbb{P}(X)$. Moreover, given $x \in X$, we see that $b_l \leq \llbracket Tx = \tau x = \lambda_0 x \rrbracket \wedge \llbracket \lambda_0 = \lambda_l^\wedge \rrbracket \leq \llbracket Tx = \lambda_l^\wedge x \rrbracket$, so that $P_l Tx = P_l(\lambda_l x) = \lambda_l P_l(x)$. Summing up over $l = 1, \dots, m$, we get $Tx = \lambda_1 P_1 x + \dots + \lambda_m P_m x$.

(6) \implies (1) Arguing for a contradiction, assume that 2.3.6(2) is fulfilled and construct a nonzero band preserving nilpotent operator in X . By 2.2.7 and 2.1.6 we have $\mathbb{V}^{(\mathbb{B})} \models \mathcal{R} \neq \mathbb{R}^\wedge$ and thus \mathcal{R} is an infinite-dimensional vector space over \mathbb{R}^\wedge within $\mathbb{V}^{(\mathbb{B})}$. Let $\mathcal{E} \subset \mathcal{R}$ be a Hamel basis and choose an infinite sequence $(e_n)_{n \in \mathbb{N}}$ of pairwise distinct elements in \mathcal{E} . Fix a natural $m > 1$ and define an \mathbb{R}^\wedge -linear function $\tau : \mathcal{R} \rightarrow \mathcal{R}$ within $\mathbb{V}^{(\mathbb{B})}$ by letting $\tau(e_{km+i}) = e_{km+i-1}$ if $2 \leq i \leq m$, $\tau(e_{km+1}) = 0$ for all $k := 0, 1, \dots$, and $\tau(e) = 0$ if $e \neq e_n$ for all $n \in \mathbb{N}$. In other words, if \mathcal{R}_0 is the \mathbb{R}^\wedge -linear subspace of \mathcal{R} generated by the sequence $(e_n)_{n \in \mathbb{N}}$, then \mathcal{R}_0 is an invariant subspace for τ and τ is the linear operator associated to the infinite block matrix $\text{diag}(A, \dots, A, \dots)$ with equal blocks in the principal diagonal and A a the Jordan block of size m with eigenvalue 0. It follows that τ is discontinuous and $\tau^m = 0$ by construction. Consequently, $T := \tau\downarrow$ is a band preserving linear operator in X and $T^m = 0$ by 2.3.3, but T is not order bounded; a contradiction. ▷

2.3.7. Algebraic order bounded disjointness preserving operators in vector lattices were treated in [14] where, in particular, the Propositions 2.3.2 and 2.3.5 were proved. Theorem 2.3.6 was obtained in [53].

2.4. Involutions and Complex Structures

The main result of this section tells us that in a real non locally one-dimensional universally complete vector lattice there are band preserving complex structures and nontrivial band preserving involutions.

2.4.1. A linear operator T on a vector lattice X is called *involutory* or an *involution* if $T \circ T = I_X$ (or, equivalently, $T^{-1} = T$) and is called a *complex structure* if $T \circ T = -I_X$ (or, equivalently, $T^{-1} = -T$). The operator $P - P^\perp$, where P is a projection operator on X and $P^\perp = I_X - P$, is an involution. The involution $P - P^\perp$ with band projections P is referred to as *trivial*.

2.4.2. Let X be a Dedekind complete vector lattice. Then there is no order bounded band preserving complex structure in X and there is no nontrivial order bounded band preserving involution in X .

◁ An order bounded band preserving operator T on a universally complete vector lattice X with weak unit $\mathbb{1}$ is a multiplication operator: $Tx = ax$ ($x \in X$) for some $a \in X$. It follows that T is an involution if and only if $a^2 = \mathbb{1}$ and hence there is a band projection P on E with $a = P\mathbb{1} - P^\perp\mathbb{1}$ or $T = P - P^\perp$. If T is a complex structure on E then the corresponding equation $a^2 = -\mathbb{1}$ has no solution. ▷

2.4.3. Theorem. *Let \mathbb{F} be a dense subfield of \mathbb{R} and $B \subset \mathbb{R}$ be a non-empty finite or countable set. Then there exists a discontinuous \mathbb{F} -linear function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \circ f = \text{id}$ and $f(x) = x$ for all $x \in B$.*

◁ Let $\mathcal{E} \subset \mathbb{R}$ be a Hamel basis of \mathbb{R} over \mathbb{F} . Every $x \in B$ can be written in the form $x = \sum_{e \in \mathcal{E}} \lambda_e(x)e$, where $\lambda_e(x) \in \mathbb{F}$ for all $e \in \mathcal{E}$. Put $\mathcal{E}(x) := \{e \in \mathcal{E} : \lambda_e(x) \neq 0\}$ and $\mathcal{E}_0 = \bigcup_{x \in B} \mathcal{E}(x)$. Since B is finite or countable, so is also \mathcal{E}_0 . Hence $\mathcal{E} \setminus \mathcal{E}_0$ has the cardinality of continuum. There exists a decomposition $\mathcal{E}_1 \cup \mathcal{E}_2 = \mathcal{E} \setminus \mathcal{E}_0$, where \mathcal{E}_1 and \mathcal{E}_2 disjoint sets both having the same cardinality. Hence there exists a one-to-one mapping g_0 from \mathcal{E}_1 onto \mathcal{E}_2 with the inverse $g_0^{-1} : \mathcal{E}_2 \rightarrow \mathcal{E}_1$.

Now we define the function $g : \mathbb{R} \rightarrow \mathbb{R}$ as follows:

$$g(h) = \begin{cases} g_0(h), & \text{for } h \in \mathcal{E}_1, \\ g_0^{-1}(h), & \text{for } h \in \mathcal{E}_2, \\ h, & \text{for } h \in \mathcal{E}_0. \end{cases} \quad (1)$$

It can be easily checked that the \mathbb{F} -linear extension $f : \mathbb{R} \rightarrow \mathbb{R}$ of a function g is the sought involution. ▷

2.4.4. Theorem. *Let \mathbb{F} be a dense subfield of \mathbb{R} . Then there exists a discontinuous \mathbb{F} -linear function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \circ f = -\text{id}$.*

◁ The proof is similar to that of Theorem 4.13.3 with the minor modifications: put $\mathcal{E}_0 = \emptyset$ and define

$$g(h) = \begin{cases} -g_0(h), & \text{for } h \in \mathcal{E}_1, \\ g_0^{-1}(h), & \text{for } h \in \mathcal{E}_2. \end{cases} \quad \triangleright$$

Interpreting Theorems 2.4.3 and 2.4.4 in a Boolean valued model yields the result.

2.4.5. Theorem. *Let X be a universally complete real vector lattice that is not locally one-dimensional. Then*

(1) *For every nonempty finite or countable set $B \subset X$ there exists a band preserving involution T on X with $T(x) = x$ for all $x \in B$.*

(2) *There exists a band preserving complex structure on X .*

◁ Assume that $X = \mathcal{R} \downarrow$. Take a one-to-one function $\nu : \mathbb{N} \rightarrow X$ with $B = \text{Im}(\nu)$. The function $\nu \uparrow : \mathbb{N}^\wedge \rightarrow X$ may fail to be one-to-one within $\mathbb{V}^{(\mathbb{B})}$ but $B \uparrow$ is again finite or countable, as $B \uparrow = \text{Im}(\nu \uparrow)$ by 1.2.7. By Theorem 2.4.3

there exists an \mathbb{R}^\wedge -linear function $\tau : \mathcal{R} \rightarrow \mathcal{R}$ such that $\llbracket \tau \circ \tau = I_{\mathcal{R}} \rrbracket = 1$ and

$$\begin{aligned} 1 &= \llbracket (\forall x \in B^\uparrow) \tau(x) = x \rrbracket = \llbracket (\forall n \in \mathbb{N}^\wedge) \tau(\nu^\uparrow(n)) = \nu^\uparrow(n) \rrbracket \\ &= \bigwedge_{n \in \mathbb{N}} \llbracket \tau(\nu^\uparrow(n^\wedge)) = \nu^\uparrow(n^\wedge) \rrbracket = \bigwedge_{n \in \mathbb{N}} \llbracket \tau(\nu(n)) = \nu(n) \rrbracket \\ &= \bigwedge_{n \in \mathbb{N}} \llbracket \tau \downarrow(\nu(n)) = \nu(n) \rrbracket. \end{aligned}$$

It follows that if $T := \tau \downarrow$ then $T \circ T = I_X$ by 1.2.4 and $T(\nu(n)) = \nu(n)$ for all $n \in \mathbb{N}$ as required in (1). The second claim is proved in a similar way on using Theorem 2.4.4. \triangleright

2.4.6. Corollary. *Let X be a universally complete vector lattice. Then the following are equivalent:*

- (1) X is locally one-dimensional.
- (2) There is no nontrivial band preserving involution on X .
- (3) There is no band preserving complex structure on X .

2.4.7. Corollary. *Let X be a universally complete real vector lattice. Then X admits a structure of complex vector space with a band preserving complex multiplication.*

\triangleleft A complex structure T on X allows to define on X a structure of a vector space over the complexes \mathbb{C} , by setting $(\alpha + i\beta)x = \alpha x + \beta T(x)$ for all $z = \alpha + i\beta \in \mathbb{C}$ and $x \in E$. If T is band preserving then the map $x \mapsto zx$ ($x \in E$) is evidently band preserving for all $z \in \mathbb{C}$. \triangleright

2.4.8. The main results of this section were obtained in [54]. In connection with Corollary 2.4.7 spaces without complex structure should be mentioned, see [18, 26, 27, 75].

Chapter 3. Disjointness Preserving Operators

3.1. Characterization and Representation

Now we will demonstrate that some properties of disjoint preserving operators are just Boolean valued interpretations of elementary properties of disjointness preserving functionals.

3.1.1. Theorem. *Assume that Y has the projection property. An order bounded linear operator $T : X \rightarrow Y$ is disjointness preserving if and only if $\ker(bT)$ is an order ideal in X for every projection $b \in \mathbb{P}(Y)$.*

\triangleleft The necessity is obvious, and so only the sufficiency will be proved. Suppose that $\ker(bT)$ is an order ideal in X for every $b \in \mathbb{P}(Y)$. We can assume that $Y \subset \mathcal{R} \downarrow$ by the Gordon Theorem. Let $|y| \leq |x|$ and $b := \llbracket Tx = 0 \rrbracket$. Then $bTx = 0$ by (G) and $bTy = 0$ by the hypothesis. Again, making use of (G) we have $b \leq \llbracket Ty = 0 \rrbracket$. Thus $\llbracket Tx = 0 \rrbracket \leq \llbracket Ty = 0 \rrbracket$ or, what is the same, $\llbracket Tx = 0 \rrbracket \Rightarrow \llbracket Ty = 0 \rrbracket = 1$. Now, put $\tau := T \uparrow$ and ensure that $\ker(\tau)$ is an order

ideal in X^\wedge within $V^{(\mathbb{B})}$. Making use of the fact that $|x| \leq |y|$ if and only if $\llbracket x^\wedge \leq y^\wedge \rrbracket = 1$, we deduce:

$$\begin{aligned} & \llbracket \ker(\tau) \text{ is an order ideal in } X^\wedge \rrbracket \\ &= \llbracket (\forall x, y \in X^\wedge) (\tau(x) = 0 \wedge |y| \leq |x| \rightarrow \tau(y) = 0) \rrbracket \\ &= \bigwedge_{x, y \in X} \llbracket (\tau(x^\wedge) = 0) \wedge \llbracket |y^\wedge| \leq |x^\wedge| \rrbracket \Rightarrow \llbracket \tau(y^\wedge) = 0 \rrbracket \\ &= \bigwedge \left\{ \llbracket T(x) = 0 \rrbracket \Rightarrow \llbracket T(y) = 0 \rrbracket : x, y \in X, |y| \leq |x| \right\} = 1. \end{aligned}$$

Apply within $V^{(\mathbb{B})}$ the fact that the functional τ is disjointness preserving if and only if $\ker(\tau)$ is an order ideal in X^\wedge . It follows that T is also disjointness preserving according to 1.4.4 (5). \triangleright

3.1.2. Similar reasoning yields that if Y has the projection property then for an order bounded disjointness preserving linear operator $T \in L^\sim(X, Y)$ there exists a band projection $\pi \in \mathbb{P}(Y)$ such that $T^+ = \pi|T|$ and $T^- = \pi^\perp|T|$. In particular, $T = (\pi - \pi^\perp)|T|$ and $|T| = (\pi - \pi^\perp)T$. To ensure this, observe that the functional $\tau := T \upharpoonright$ preserves disjointness if and only if either τ , or $-\tau$ is a lattice homomorphism.

From this fact it follows that $T \in L^\sim(X, Y)$ is disjointness preserving if and only if $(Tx)^+ \perp (Ty)^-$ for all $x, y \in X_+$. Indeed, for arbitrary $x, y \in X_+$ we can write $(Tx)^+ = (Tx) \vee 0 \leq T^+x = \pi|T|x$ and, similarly, $(Ty)^- \leq \pi^\perp|T|y$. Hence $(Tx)^+ \wedge (Ty)^- = 0$.

3.1.3. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. For a pair of disjointness preserving operators T_1 and T_2 from X to Y there exist a band projection $\pi \in \mathbb{P}(Y)$, a lattice homomorphism $T \in \text{Hom}(X, Y)$, and orthomorphisms $S_1, S_2 \in \text{Orth}(Y)$ such that*

$$\begin{aligned} & |S_1| + |S_2| = \pi, \quad \pi T_1 = S_1 T, \quad \pi T_2 = S_2 T, \\ & \text{Im}(\pi^\perp T_1)^{\perp\perp} = \text{Im}(\pi^\perp T_2)^{\perp\perp} = \pi(Y), \quad \pi^\perp T_1 \perp \pi^\perp T_2. \end{aligned}$$

\triangleleft As usual, there is no loss of generality in assuming that $Y = \mathcal{R} \downarrow$. Put $\tau_1 := T_1 \upharpoonright$ and $\tau_2 := T_2 \upharpoonright$. The desired result is a Boolean valued interpretation of the following fact: If the disjointness preserving functionals τ_1 and τ_2 are not proportional then they are nonzero and disjoint. Put $b := \llbracket \tau_1 \text{ and } \tau_2 \text{ are proportional} \rrbracket$ and $\pi := \chi(b)$. Then within $V^{(\{0, b\})}$ there exist a lattice homomorphism $\tau : X^\wedge \rightarrow \mathcal{R}$ and reals $\sigma_1, \sigma_2 \in \mathcal{R}$ such that $\tau_i = \sigma_i \tau$. If the function $\bar{\sigma}_i$ is defined as $\bar{\sigma}_i(\lambda) := \sigma_i \lambda$ ($\lambda \in \mathcal{R}$), then the operators $S_1 := \sigma_1 \downarrow$, $S_2 := \sigma_2 \downarrow$, and $T := \tau \downarrow$ satisfy the first line of required conditions. Moreover, $b^* = \llbracket \tau_1 \neq 0 \rrbracket \wedge \llbracket \tau_2 \neq 0 \rrbracket \wedge \llbracket \tau_1 \wedge \tau_2 = 0 \rrbracket$ and $\pi^\perp = \chi(b^*)$, so that the second line of required conditions is also satisfied. \triangleright

3.1.4. Corollary. *Let X and Y be vector lattices with Y Dedekind complete. The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2 : X \rightarrow Y$ is disjointness preserving if and only if there exist pairwise disjoint band projections $\pi, \pi_1, \pi_2 \in \mathbb{P}(Y)$, orthomorphisms $S_1, S_2 \in \text{Orth}(Y)$ and a lattice*

homomorphism $T \in \text{Hom}(X, Y)$ such that the following system of relations is consistent

$$\begin{aligned}\pi + \pi_1 + \pi_2 &= I_Y, \quad |S_1| + |S_2| = \pi, \\ T(X)^{\perp\perp} &= \pi(Y), \quad \pi_1 T_2 = \pi_2 T_1 = 0, \\ \pi T_1 &= S_1 T, \quad \pi T_2 = S_2 T.\end{aligned}$$

Consequently, in this case $T_1 + T_2 = \pi_1 T_1 + \pi_2 T_2 + (S_1 + S_2)T$.

3.1.5. Corollary. *The sum $T_1 + T_2$ of two disjointness preserving operators $T_1, T_2 : X \rightarrow Y$ is disjointness preserving if and only if $T_1(x_1) \perp T_2(x_2)$ for all $x_1, x_2 \in X$ with $x_1 \perp x_2$.*

\triangleleft The necessity is immediate from Theorem 3.1.4, since $T_1 = \pi_1 T_1 + S_1 T$ and $T_2 = \pi_2 T_2 + S_2 T$. To see the sufficiency, observe that if T_1 and T_2 meet the above condition then $T_k x_1 \perp T_l x_2$ ($k, l := 1, 2$) and thus $(T_1 + T_2)(x_1) \perp (T_1 + T_2)(x_2)$ for any pair of disjoint elements $x_1, x_2 \in X$. \triangleright

3.1.6. Aspects of the theory of disjointness preserving operators are presented in [30, 41, 42]. The recent results on disjointness preserving operators are surveyed in [13]. In particular, the concept of disjointness preserving set of operators is discussed in the survey. Using this concept Corollary 3.1.5 may be reformulated as follows: $T_1 + T_2$ is disjointness preserving if and only if $\{T_1, T_2\}$ is a disjointness preserving set of operators.

3.2. Polydisjoint Operators

The aim of the present section is to describe the order ideal in the space of order bounded operators generated by order bounded disjointness preserving operators (= d -homomorphisms) in terms of n -disjoint operators.

3.2.1. Let X and Y be vector lattices and n be a positive integer. A linear operator $T : X \rightarrow Y$ is said to be n -disjoint if, for any collection of $n + 1$ pairwise disjoint elements $x_0, \dots, x_n \in X$, the infimum of the set $\{|Tx_k| : k := 0, 1, \dots, n\}$ equals zero; symbolically:

$$(\forall x_0, x_1, \dots, x_n \in X) \ x_k \perp x_l \ (k \neq l) \implies |Tx_0| \wedge \dots \wedge |Tx_n| = 0.$$

An operator is called *polydisjoint* if it is n -disjoint for some $n \in \mathbb{N}$. A 1-disjoint operator is just a disjointness preserving operator.

3.2.2. Consider some simple properties of n -disjoint operators. Let X and Y be vector lattices with Y Dedekind complete.

(1) An operator $T \in L^\sim(X, Y)$ is n -disjoint if and only if $|T|$ is n -disjoint.

(2) Let T_1, \dots, T_n be order bounded and disjointness preserving operators from X to Y . Then the operator $T := T_1 + \dots + T_n$ is n -disjoint.

3.2.3. *An order bounded functional on a vector lattice is n -disjoint if and only if it is representable as a disjoint sum of n order bounded disjointness preserving functionals. Such representation is unique up to permutation.*

\triangleleft Assume that f is a positive n -disjoint functional on a vector lattice $C(Q)$. Prove that the corresponding Radon measure μ is the sum of n Dirac measures.

This is equivalent to saying that the support of μ consists of n points. If there are $n + 1$ points $q_0, q_1, \dots, q_n \in Q$ in the support of μ then we may choose pair-wise disjoint compact neighborhoods $U_0, U_1, \dots, U_n \in Q$ of these points and next take pair-wise disjoint open sets $V_k \subset Q$ with $\mu(U_k) > 0$ and $U_k \subset V_k$ ($k = 0, 1, \dots, n$). Using the Tietze–Urysohn Theorem construct a continuous function x_k on Q which vanishes on $Q \setminus V_k$ and is identically equal to 1 on U_k . Then $x_0 \wedge x_1 \wedge \dots \wedge x_n = 0$ but none of $f(x_0), f(x_1), \dots, f(x_n)$ is equal to zero, since $f(x_k) \geq \mu(U_k) > 0$ for all $k := 0, 1, \dots, n$. This contradiction shows that the support of μ consists of n points. The general case is reduced to what was proven by using the Kreĭns–Kakutani Representation Theorem. \triangleright

3.2.4. Theorem. *An order bounded operator from a vector lattice to a Dedekind complete vector lattice is n -disjoint for some $n \in \mathbb{N}$ if and only if it is representable as a disjoint sum of n order bounded disjointness preserving operators.*

\triangleleft Assume that the operator $T \in L^\sim(X, Y)$ is n -disjoint and denote $\tau := T\uparrow \in \mathbb{V}^{(\mathbb{B})}$. It is deduced by direct calculation of Boolean truth values that $\tau : X^\wedge \rightarrow \mathcal{R}$ is an order bounded n -disjoint functional within $\mathbb{V}^{(\mathbb{B})}$. According to Transfer Principle, applying Proposition 3.2.3 to τ yields pairwise disjoint order bounded disjointness preserving functionals τ_1, \dots, τ_n on X^\wedge with $\tau = \tau_1 + \dots + \tau_n$. It remains to observe that the linear operators $T_1 := \tau_1\downarrow, \dots, T_n := \tau_n\downarrow$ from X to Y are order bounded, disjointness preserving, and $T_1 + \dots + T_n = T$. Moreover, if $k \neq j$ then $0 = (\tau_k \wedge \tau_j)\downarrow = \tau_k\downarrow \wedge \tau_j\downarrow = T_k \wedge T_j$, so that T_k and T_j are disjoint. \triangleright

3.2.5. It can be easily seen that the representation of an order bounded n -disjoint operator in Theorem 3.1.4 is unique up to mixing: if $T = T_1 + \dots + T_n = S_1 + \dots + S_m$ for two pairwise disjoint collections $\{T_1, \dots, T_n\}$ and $\{S_1, \dots, S_m\}$ of order bounded disjointness preserving operators then for every $j = 1, \dots, m$ there exists a disjoint family of projections $\pi_{1j}, \dots, \pi_{nj} \in \mathbb{P}(Y)$ such that $S_j = \pi_{1j}T_1 + \dots + \pi_{nj}T_n$ for all $j := 1, \dots, m$.

3.2.6. Corollary. *A positive operator from a vector lattice to a Dedekind complete vector lattice is n -disjoint if and only if it is the sum of n lattice homomorphisms.*

3.2.7. Corollary. *The set of polydisjoint operators from a vector lattices to a Dedekind complete vector lattices coincides with the order ideal in the vector lattice of order bounded operators generated by lattice homomorphisms.*

3.2.8. The characterizations of sums of lattice homomorphisms (Corollary 3.2.6), sums of disjointness preserving operators (Theorem 3.2.4), and the ideal of order bounded operators generated by lattice homomorphism (Corollary 3.2.7) were proved in [12] using standard tools. An algebraic approach to the problem see in [41].

3.3. Differences of Lattice Homomorphisms

This section presents a characterization of order bounded operators representable as a difference of two lattice homomorphisms. The starting point of this question is the celebrated Stone Theorem about the structure of vector sub-

lattices in the Banach lattice $C(Q, \mathbb{R})$ of continuous real functions on a compact space Q . This theorem may be rephrased in the above terms as follows:

3.3.1. Stone Theorem. *Each closed vector sublattice of $C(Q, \mathbb{R})$ is the intersection of the kernels of some differences of lattice homomorphisms on $C(Q, \mathbb{R})$.*

3.3.2. In view of this theorem it is reasonable to refer to a difference of lattice homomorphisms on a vector lattice X as a two-point relation on X . We are not obliged to assume here that the lattice homomorphisms under study act into the reals \mathbb{R} . Thus a linear operator $T : X \rightarrow Y$ between vector lattices is said to be a *two-point relation* on X whenever it is written as a difference of two lattice homomorphisms. An operator $bT := b \circ T$ with $b \in \mathbb{B} := \mathbb{P}(Y)$ is called a *stratum* of T .

3.3.3. The kernel $\ker(bT)$ of any stratum of a two point relation T is evidently a sublattice of X , since it is determined by an equation $bT_1x = bT_2x$. Thus, each stratum bT of an order bounded disjointness preserving operator $T : X \rightarrow Y$ is a two-point relation on X and so its kernel is a vector sublattice of X . The main result of this section says that the converse is valid too. To handle the corresponding scalar problem a formula of subdifferential calculus is used; cp. [56, 46]. In the following form of this auxiliary fact *positive decomposition* of a functional f means any representation $f = f_1 + \dots + f_N$ with positive functionals f_1, \dots, f_N .

3.3.4. Decomposition Theorem. *Assume that H_1, \dots, H_N are cones in a vector lattice X and f and g are positive functionals on X . The inequality*

$$f(h_1 \vee \dots \vee h_N) \geq g(h_1 \vee \dots \vee h_N)$$

holds for all $h_k \in H_k$ ($k := 1, \dots, N$) if and only if to each positive decomposition (g_1, \dots, g_N) of g there is a positive decomposition (f_1, \dots, f_N) of f such that

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k; k := 1, \dots, N).$$

3.3.5. Let \mathbb{F} be a dense subfield in \mathbb{R} and X a vector lattice over \mathbb{F} . An order bounded \mathbb{F} -linear functional from X to \mathbb{R} is a two-point relation if and only if its kernel is a \mathbb{F} -linear sublattice of the ambient vector lattice.

◁ Let l be an order bounded functional on a vector lattice X . Denote $f := l^+$, $g := l^-$, and $H := \ker(l)$. It suffices to demonstrate only that g is a lattice homomorphism, i. e., $[0, g] = [0, 1]g$; cp. [41]. So, we take $0 \leq g_1 \leq g$ and put $g_2 := g - g_1$. We may assume that $g_1 \neq 0$ and $g_1 \neq g$. By hypothesis, for all $h_1, h_2 \in \ker(l)$ we have the $f(h_1 \vee h_2) \geq g(h_1 \vee h_2)$. By the Decomposition theorem there is a positive decomposition $f = f_1 + f_2$ such that $f_1(h) - g_1(h) = 0$ and $f_2(h) - g_2(h) = 0$ for all $h \in H$. Since $H = \ker(f - g)$, we see that there are reals α and β satisfying $f_1 - g_1 = \alpha(f - g)$ and $f_2 - g_2 = \beta(f - g)$. Clearly, $\alpha + \beta = 1$ (for otherwise $f = g$ and $l = 0$). Therefore, one of the reals α and β is strictly positive. If $\alpha > 0$ then we have $g_1 = \alpha g$ for f and g are disjoint. If $\beta > 0$ then, arguing similarly, we see that $g_2 = \beta g$. Hence, $0 \leq \beta \leq 1$ and we again see that $g_1 \in [0, 1]g$. ▷

3.3.4. Theorem. *An order bounded operator from a vector lattice to a Dedekind complete vector lattice is a two-point relation if and only if the kernel of its every stratum is a vector sublattice of the ambient vector lattice.*

◁ The necessity is obvious, so only the sufficiency will be proved. Let $T \in L^\sim(X, Y)$ and $\ker(bT) := (bT)^{-1}(0)$ is a vector sublattice in X for all $b \in \mathbb{P}(Y)$. We apply the Boolean valued “scalarization” putting $Y = \mathcal{R}\downarrow$.

Denote $\tau := T\uparrow$ and observe that the validity of identities $T^+\uparrow = \tau^+$ and $T^-\uparrow = \tau^-$ within $\mathbb{V}(\mathbb{B})$ is proved by easy calculation of Boolean truth values. Moreover, $\llbracket \ker(\tau) \text{ is a vector sublattice of } X^\wedge \rrbracket = 1$. Indeed, given $x, y \in X$, put $b := \llbracket Tx = 0^\wedge \rrbracket \wedge \llbracket Ty = 0^\wedge \rrbracket$. This means that $x, y \in \ker(bT)$. Hence, we see by hypothesis that $bT(x \vee y) = 0$, whence $b \leq \llbracket T(x \vee y) = 0^\wedge \rrbracket$. Replacing T by τ yields $\llbracket \tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \rrbracket \leq \llbracket \tau(x \vee y)^\wedge = 0^\wedge \rrbracket$. A straightforward calculation of Boolean truth values completes the proof:

$$\begin{aligned} & \llbracket \ker(\tau) \text{ is a Riesz subspace of } X^\wedge \rrbracket \\ &= \llbracket (\forall x, y \in X^\wedge)(\tau(x) = 0^\wedge \wedge \tau(y) = 0^\wedge \rightarrow \tau(x \vee y) = 0^\wedge) \rrbracket \\ &= \bigwedge_{x, y \in X} \llbracket \tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \rightarrow \tau((x \vee y)^\wedge) = 0^\wedge \rrbracket = 1. \triangleright \end{aligned}$$

3.3.7. Theorems 3.3.4 and 3.3.6 were obtained in [55] and [57], respectively. On using of the above terminology, the Meyer Theorem (see [41, 3.3.1 (5)] and [21, 68]) reads as follows: *Each order bounded disjointness preserving operator between vector lattices is a two-point relation.* This fact can be easily deduced from Theorem 3.3.6, since $\ker(bT)$ is a vector sublattice, whenever T is disjointness preserving.

3.4. Sums of Lattice Homomorphisms

In this section we will give a description for an order bounded operator T whose modulus may be presented as the sum of two lattice homomorphisms in terms of the properties of the kernels of the strata of T . Thus, we reveal the connection between the 2-disjoint operators and Grothendieck subspaces.

3.4.1. Recall that a subspace H of a vector lattice is a *G-space* or *Grothendieck subspace* provided that H enjoys the following property:

$$(\forall x, y \in H) (x \vee y \vee 0 + x \wedge y \wedge 0 \in H). \quad (1)$$

3.4.2. This condition appears as follows. In 1955 Grothendieck [28] pointed out the subspaces with the above condition in the vector lattice $C(Q, \mathbb{R})$ of continuous functions on a compact space Q defining them by means of a family of relations A with each relation $\alpha \in A$ being the form:

$$f(q_\alpha^1) = \lambda_\alpha f(q_\alpha^2) \quad (q_\alpha^1, q_\alpha^2 \in Q; \lambda_\alpha \in \mathbb{R}; \alpha \in A).$$

These spaces yield examples of L^1 -predual Banach spaces which are not *AM*-spaces. In 1969 Lindenstrauss and Wulpert gave a characterization of such

subspaces by means of the property 3.4.1 and introduced the term G -space (see [60]). Some related properties of Grothendieck spaces are presented also in [59] and [72].

3.4.3. Theorem. *Let \mathbb{F} be a dense subfield in \mathbb{R} and X a vector lattice over \mathbb{F} . The modulus of an order bounded \mathbb{F} -linear functional from X to \mathbb{R} is the sum of two lattice homomorphisms if and only if the kernel of this functional is a Grothendieck subspace of X .*

◁ The proof relied on Decomposition Theorem 3.3.4; cp. [58]. ▷

3.4.4. Theorem. *Let X and Y be vector lattices with Y Dedekind complete. The modulus of an order bounded operator $T : X \rightarrow Y$ is the sum of some pair of lattice homomorphisms if and only if the kernel of each stratum bT of T with $b \in \mathbb{B} := \mathbb{P}(Y)$ is a Grothendieck subspace of the ambient vector lattice X .*

◁ The proof runs along the lines of § 3.3. We apply the technique of Boolean valued “scalarization” reducing operator problems to the case of functionals handled in 3.4.3. Put $Y = \mathcal{R}\downarrow$ and $\tau := T\uparrow$ and the further work with τ is performed within $\mathbb{V}^{(\mathbb{B})}$. First we observe a useful calculation:

$$\begin{aligned} & \llbracket \ker(l) \text{ is a Grothendieck subspace of } X^\wedge \rrbracket \\ &= \llbracket (\forall x, y \in X^\wedge) (\tau(x) = 0^\wedge \wedge \tau(y) = 0^\wedge \rightarrow \tau(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge) \rrbracket \\ &= \bigwedge_{x, y \in X} \llbracket \tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \rightarrow \tau((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge \rrbracket. \quad (*) \end{aligned}$$

Sufficiency: Take $x, y \in X$ and put $b := \llbracket Tx = 0^\wedge \rrbracket \wedge \llbracket Ty = 0^\wedge \rrbracket$. It follows from (G) (see the Gordon Theorem) that $x, y \in \ker(bT)$. By hypothesis $\ker(bT)$ is a Grothendieck subspace and hence $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. By using (G) again we get

$$\llbracket Tx = 0^\wedge \rrbracket \wedge \llbracket Ty = 0^\wedge \rrbracket = b \leq \llbracket T(x \vee y \vee 0 + x \wedge y \wedge 0) = 0^\wedge \rrbracket.$$

Now, it follows from (*) that $\llbracket \ker(l) \text{ is a Grothendieck subspace of } X^\wedge \rrbracket = 1$. By Transfer Principle we can apply Theorem 3.4.3 to τ within $\mathbb{V}^{(\mathbb{B})}$, consequently, $|\tau| = \tau_1 + \tau_2$ with τ_1 and τ_2 being lattice homomorphisms within $\mathbb{V}^{(\mathbb{B})}$. It can be easily seen that the operators $T_1 := \tau_1\downarrow$ and $T_2 := \tau_2\downarrow$ from X to $\mathcal{R}\downarrow$ are lattice homomorphisms and $|T| = T_1 + T_2$.

Necessity: Assume that $|T| = T_1 + T_2$ for some lattice homomorphisms $T_1, T_2 : X \rightarrow Y$ and denote $\tau := T\uparrow$, $\tau_1 := T_1\uparrow$ and $\tau_2 := T_2\uparrow$. It can be easily checked that inside $\mathbb{V}^{(\mathbb{B})}$ we have $\tau, \tau_1, \tau_2 : X^\wedge \rightarrow \mathcal{R}$ and $|\tau| = \tau_1 + \tau_2$; moreover, τ_1 and τ_2 lattice homomorphisms. By Theorem 3.4.3 and Transfer Principle $\llbracket \ker(l) \text{ is a Grothendieck subspace of } X^\wedge \rrbracket = 1$. Making use of (*), we infer

$$\llbracket \tau(x^\wedge) = 0^\wedge \wedge \tau(y^\wedge) = 0^\wedge \rrbracket \leq \llbracket \tau((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge \rrbracket.$$

Now, if $b \in \mathbb{B}$ and $bTx = bTy = 0$ then $\llbracket l((x \vee y \vee 0 + x \wedge y \wedge 0)^\wedge) = 0^\wedge \rrbracket \geq b$, whence by the Gordon Theorem we get $bT(x \vee y \vee 0 + x \wedge y \wedge 0) = 0$. ▷

3.4.5. The main result of the section (Theorem 3.4.4) was obtained in [58]. The sums of Riesz homomorphisms were first described in [12] in terms of

n -disjoint operators; see § 3.3. A survey of some conceptually close results on n -disjoint operators is given in [3, § 5.6].

3.5. Disjointness Preserving Bilinear Operators

It was observed above that a linear operator T from a vector lattice X to a Dedekind complete vector lattice Y is in a sense determined up to an orthomorphism from the family of the kernels of the strata πT of T with π ranging over all band projections on Y . Similar reasoning was involved in [50] to characterize order bounded disjointness preserving bilinear operators. Unfortunately, Theorem 3.4 in [50] is erroneous and this note aims to give the correct statement and proof of this result.

In what follows X , Y , and Z are Archimedean vector lattices, Z^u is a universal completion of Z , and $B : X \times Y \rightarrow Z$ is a bilinear operator. We denote the Boolean algebra of band projections in X by $\mathbb{P}(X)$. Recall that a linear operator $T : X \rightarrow Y$ is said to be *disjointness preserving* if $x \perp y$ implies $Tx \perp Ty$ for all $x, y \in X$. A bilinear operator $B : X \times Y \rightarrow Z$ is called *disjointness preserving* (a *lattice bimorphism*) if the linear operators $B(x, \cdot) : y \mapsto B(x, y)$ ($y \in Y$) and $B(\cdot, y) : x \mapsto B(x, y)$ ($x \in X$) are disjointness preserving for all $x \in X$ and $y \in Y$ (lattice homomorphisms for all $x \in X_+$ and $y \in Y_+$). Denote $X_\pi := \bigcap \{\ker(\pi B(\cdot, y)) : y \in Y\}$ and $Y_\pi := \bigcap \{\ker(\pi B(x, \cdot)) : x \in X\}$. Clearly, X_π and Y_π are vector subspaces of X and Y , respectively. Now we state the main result of the note.

3.5.1. Theorem. *Assume that X , Y , and Z are vector lattices with Z having the projection property. For an order bounded bilinear operator $B : X \times Y \rightarrow Z$ the following are equivalent:*

- (1) B is disjointness preserving.
- (2) *There are a band projection $\varrho \in \mathbb{P}(Z)$ and lattice homomorphisms $S : X \rightarrow Z^u$ and $T : Y \rightarrow Z^u$ such that $B(x, y) = \varrho S(x)T(y) - \varrho^\perp S(x)T(y)$ for all $(x, y) \in X \times Y$.*
- (3) *For every $\pi \in \mathbb{P}(Z)$ the subspaces X_π and Y_π are order ideals respectively in X and Y , and the kernel of every stratum πB of B with $\pi \in \mathbb{P}(Z)$ is representable as*

$$\ker(\pi B) = \bigcup \{X_\sigma \times Y_\tau : \sigma, \tau \in \mathbb{P}(Z); \sigma \vee \tau = \pi\}.$$

The proof proceeds along the general lines of [57]–[50]: Using the canonical embedding and ascent to the Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$, we reduce the matter to characterizing a disjointness preserving bilinear functional on the product of two vector lattices over a dense subfield of the reals \mathbb{R} . The resulting scalar problem is solved by the following simple fact.

3.5.2 *Let X and Y be vector lattices. For an order bounded bilinear functional $\beta : X \times Y \rightarrow \mathbb{R}$ the following are equivalent:*

- (1) β is disjointness preserving.

(2) $\ker(\beta) = (X_0 \times Y) \cup (X \times Y_0)$ for some order ideals $X_0 \subset X$ and $Y_0 \subset Y$.

(3) There exist lattice homomorphisms $g : X \rightarrow \mathbb{R}$ and $h : Y \rightarrow \mathbb{R}$ such that either $\beta(x, y) = g(x)h(y)$ or $\beta(x, y) = -g(x)h(y)$ for all $x \in X$ and $y \in Y$.

◁ Assume that $\ker(\beta) = (X_0 \times Y) \cup (X \times Y_0)$ and take $y \in Y$. If $y \in Y_0$ then $\beta(\cdot, y) \equiv 0$, otherwise $\ker(\beta(\cdot, y)) = X_0$ and $\beta(\cdot, y)$ is disjointness preserving, since an order bounded linear functional is disjointness preserving if and only if its null-space is an order ideal. Similarly, $\beta(x, \cdot)$ is disjointness preserving for all $x \in X$ and thus (2) \implies (1). The implication (1) \implies (3) was established in [52, Theorem 3.2] and (3) \implies (1) is trivial with $X_0 = \ker(g)$ and $Y_0 = \ker(h)$. ▷

Let \mathbb{B} be a complete Boolean algebra and let $\mathbb{V}^{(\mathbb{B})}$ be the corresponding Boolean valued model with the Boolean truth value $\llbracket \varphi \rrbracket$ of a set-theoretic formula φ . There exists $\mathcal{R} \in \mathbb{V}^{(\mathbb{B})}$ playing the role of the field of reals within $\mathbb{V}^{(\mathbb{B})}$. The descent functor sends each internal algebraic structure \mathfrak{A} into its descent $\mathfrak{A} \downarrow$ which is an algebraic structure in the conventional sense. Gordon's theorem (see [41, 8.1.2] and [51, Theorem 2.4.2]) tells us that the algebraic structure $\mathcal{R} \downarrow$ (with the descended operations and order) is a universally complete vector lattice. Moreover, there is a Boolean isomorphism χ of \mathbb{B} onto $\mathbb{P}(\mathcal{R} \downarrow)$ such that $b \leq \llbracket x = y \rrbracket$ if and only if $\chi(b)x = \chi(b)y$. We identify \mathbb{B} with $\mathbb{P}(\mathcal{R} \downarrow)$ and take χ to be $I_{\mathbb{B}}$.

Let $[X \times Y, \mathcal{R} \downarrow] \in \mathbb{V}$ and let $\llbracket X^\wedge \times Y^\wedge, \mathcal{R} \rrbracket \in \mathbb{V}^{(\mathbb{B})}$ stand respectively for the sets of all maps from $X \times Y$ to $\mathcal{R} \downarrow$ and from $X^\wedge \times Y^\wedge$ to \mathcal{R} (within $\mathbb{V}^{(\mathbb{B})}$). The correspondences $f \mapsto f \uparrow$, the modified ascent, is a bijection between $[X \times Y, \mathcal{R} \downarrow]$ and $\llbracket X^\wedge \times Y^\wedge, \mathcal{R} \rrbracket$. Given $f \in [X, \mathcal{R} \downarrow]$, the internal map $f \uparrow \in \llbracket X^\wedge, \mathcal{R} \rrbracket$ is uniquely determined by the relation $\llbracket f \uparrow(x^\wedge) = f(x) \rrbracket = 1$ ($x \in X$). Observe also that $\pi \leq \llbracket f \uparrow(x^\wedge) = \pi f(x) \rrbracket$ ($x \in X, \pi \in \mathbb{P}(\mathcal{R} \downarrow)$). This fact specifies for bilinear operators as follows:

3.5.3. Let $B : X \times Y \rightarrow Y$ be a bilinear operator and $\beta := B \uparrow$ its modified ascent. Then $\beta : X^\wedge \times Y^\wedge \rightarrow \mathcal{R}$ is a \mathbb{R}^\wedge -bilinear functional within $\mathbb{V}^{(\mathbb{B})}$. Moreover, B is order bounded and disjointness preserving if and only $\llbracket \beta$ is order bounded and disjointness preserving $\rrbracket = 1$.

◁ The proof goes along similar lines to the proof of Theorem 3.3.3 in [51]. ▷

3.5.4. Let B and β be as in 3.4.3. Then $\llbracket \ker(B)^\wedge = \ker(\beta) \rrbracket = 1$.

◁ Using the above-mentioned property of the modified ascent and interpreting the formal definition $z \in \ker(\beta) \leftrightarrow (\exists x \in X^\wedge)(\exists y \in Y^\wedge)(z = (x, y) \wedge \beta(x, y) = 0)$, the proof reduces to the straightforward calculation:

$$\begin{aligned} \llbracket z \in \ker(\beta) \rrbracket &= \bigvee_{x \in X, y \in Y} \llbracket z = (x^\wedge, y^\wedge) \wedge \beta(x^\wedge, y^\wedge) = 0 \rrbracket \\ &= \bigvee_{(x, y) \in X \times Y} \llbracket z = (x, y)^\wedge \wedge (x, y)^\wedge \in \ker(B)^\wedge \rrbracket \end{aligned}$$

$$\begin{aligned}
\leq \llbracket z \in \ker(B)^\wedge \rrbracket &= \bigvee_{(x,y) \in X \times Y} \llbracket z = (x,y)^\wedge \wedge (x,y) \in \ker(B) \rrbracket \\
&= \bigvee_{x \in X, y \in Y} \llbracket (z = (x,y)^\wedge) \wedge \beta(x,y) = 0 \rrbracket \\
&\leq \llbracket z \in \ker(\beta) \rrbracket. \quad \triangleright
\end{aligned}$$

3.4.5. Define \mathcal{X} and \mathcal{Y} within $\mathbb{V}(\mathbb{B})$ as follows: $\mathcal{X} := \bigcap \{\ker(\beta(\cdot, Y)) : y \in Y^\wedge\}$ and $\mathcal{Y} := \bigcap \{\ker(\beta(x, \cdot)) : x \in X^\wedge\}$. Given arbitrary $\pi \in \mathbb{P}(Z)$, $x \in X$, and $y \in Y$, we have

$$\pi \leq \llbracket x^\wedge \in \mathcal{X} \rrbracket \iff x \in X_\pi, \quad \pi \leq \llbracket y^\wedge \in \mathcal{Y} \rrbracket \iff y \in Y_\pi.$$

\triangleleft Given $\pi \in \mathbb{P}(Z)$ and $x \in X$, we need only to calculate Boolean truth values taking it into account that $\llbracket B(x, y) = \beta(x^\wedge, y^\wedge) \rrbracket = 1$ for all $x \in X$ and $y \in Y$:

$$\llbracket x^\wedge \in \mathcal{X} \rrbracket = \llbracket (\forall v \in Y^\wedge) \beta(x^\wedge, v) = 0 \rrbracket = \bigwedge_{v \in Y} \llbracket \beta(x^\wedge, v^\wedge) = 0 \rrbracket = \bigwedge_{v \in Y} \llbracket B(x, v) = 0 \rrbracket$$

It follows that $\pi \leq \llbracket x^\wedge \in \mathcal{X} \rrbracket$ if and only if $\pi \leq \llbracket B(x, v) = 0 \rrbracket$ for all $v \in Y$. By Gordon's theorem the latter means that $\pi B(x, v) = 0$ for all $v \in Y$, that is $x \in X_\pi$. \triangleright

3.4.6. Let B and β be as in 3.4.3. For all $\pi \in \mathbb{P}(Z)$, $x \in X$, and $y \in Y$, we have $\pi \leq \llbracket (x^\wedge, y^\wedge) \in (\mathcal{X} \times Y) \cup (X \times \mathcal{Y}) \rrbracket$ if and only if there exist $\sigma, \tau \in \mathbb{P}(Z)$ such that $\sigma \vee \tau = \pi$, $x \in X_\sigma$, and $y \in Y_\tau$.

\triangleleft Put $\rho := \llbracket (x^\wedge, y^\wedge) \in (\mathcal{X} \times Y) \cup (X \times \mathcal{Y}) \rrbracket$ and observe that

$$\rho = \llbracket (x^\wedge \in \mathcal{X}) \vee y^\wedge \in \mathcal{Y} \rrbracket = \llbracket x^\wedge \in \mathcal{X} \rrbracket \vee \llbracket y^\wedge \in \mathcal{Y} \rrbracket.$$

Clearly, $\pi \leq \rho$ if and only if $\sigma \vee \tau = \pi$ for some $\sigma \leq \llbracket x^\wedge \in \mathcal{X} \rrbracket$ and $\tau \leq \llbracket y^\wedge \in \mathcal{Y} \rrbracket$, so that the required property follows from 3.4.5. \triangleright

PROOF OF THEOREM 3.5.1. The implication (1) \implies (2) was proved in [52, Corollary 3.3], while (2) \implies (3) is straightforward. Indeed, observe first that if (2) is fulfilled then $|B(x, y)| = |B|(|x|, |y|) = |S|(|x|)|T|(|y|)$, so that we can assume S and T to be lattice homomorphisms, as in this event $\ker(B) = \ker(|B|)$. Take $\pi \in \mathbb{P}(Z)$ and put $\sigma := \pi - \pi[Sx]$ and $\tau := \pi - \pi[Ty]$, where $[y]$ is a band projection onto $\{y\}^{\perp\perp}$. Observe next that $\pi B(x, y) = 0$ if and only if $\pi[Sx]$ and $\pi[Ty]$ are disjoint or, which is the same, if $\sigma \vee \tau = \pi$. Moreover, the map $\rho_y : x \mapsto \sigma S(x)T(y)$ is disjointness preserving for all $y \in Y$ and so $X_\sigma = \bigcap_{y \in Y} \ker(\rho_y)$ is an order ideal in X . Similarly, Y_τ is an order ideal in Y . Thus, $(x, y) \in \ker(\pi B)$ if and only if $x \in X_\sigma$ and $y \in Y_\tau$ for some $\sigma, \tau \in \mathbb{P}(Z)$ with $\sigma \vee \tau = \pi$.

Prove the remaining implication (3) \implies (1). Suppose that (3) holds for all $\pi \in \mathbb{P}(Y)$. Take $x, u \in X$ and put $\pi := \llbracket x^\wedge \in \mathcal{X} \rrbracket$ and $\rho := \llbracket |u|^\wedge \leq |x|^\wedge \rrbracket$. By 3.4.5 we have $x \in X_\pi$. Note also that either $\rho = 0$ or $\rho = 1$. If $\rho = 1$ then $|u| \leq |x|$ and by hypotheses $u \in X_\pi$. Again by 3.4.6 we get $\rho \leq \llbracket u^\wedge \in \mathcal{X} \rrbracket$. This

estimate is obvious whenever $\rho = 0$, so that $\llbracket x^\wedge \in \mathcal{X} \rrbracket \wedge \llbracket |u|^\wedge \leq |x|^\wedge \rrbracket \Rightarrow \llbracket u^\wedge \in \mathcal{X} \rrbracket = 1$ for all $x, u \in X$. Simple calculation shows now that \mathcal{X} is an order ideal in X^\wedge :

$$\begin{aligned} & \llbracket (\forall x, u \in X^\wedge)(|u| \leq |x| \wedge x \in \mathcal{X} \rightarrow u \in \mathcal{X}) \rrbracket \\ &= \bigwedge_{u, x \in X} (\llbracket x \in \mathcal{X} \rrbracket \wedge \llbracket |u| \leq |x| \rrbracket \Rightarrow \llbracket u \in \mathcal{X} \rrbracket) = 1. \end{aligned}$$

Similarly, \mathcal{Y} is an order ideal in Y^\wedge .

It follows from (3) and 3.4.6 that $(x, y) \in \ker(\pi B)$ if and only if $\pi \leq \llbracket (x^\wedge, y^\wedge) \in (\mathcal{X} \times Y) \cup (X \times \mathcal{Y}) \rrbracket$. Taking into account 3.4.3 and the observation before it we conclude that $\pi \leq \llbracket (x^\wedge, y^\wedge) \in \ker(\beta) \rrbracket$ if and only if $\pi \leq \llbracket (x^\wedge, y^\wedge) \in (\mathcal{X} \times Y) \cup (X \times \mathcal{Y}) \rrbracket$ and so $\llbracket \ker(\beta) = (\mathcal{X} \times Y) \cup (X \times \mathcal{Y}) \rrbracket = 1$. It remains to apply the equivalence (1) \iff (3) of 3.4.2 within $\mathbb{V}^{(\mathbb{B})}$. It follows that B is disjointness preserving by 3.4.3. \triangleright

3.4.7. Corollary. *Assume that Y has the projection property. An order bounded linear operator $T : X \rightarrow Y$ is disjointness preserving if and only if $\ker(bT)$ is an order ideal in X for every projection $b \in \mathbb{P}(Y)$.*

\triangleleft Apply the above theorem to the bilinear operator $B : X \times \mathbb{R} \rightarrow Y$ defined as $B(x, \lambda) = \lambda T(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}$. \triangleright

Chapter 4. Order Continuous Operators

4.1. Maharam Operators

Now we examine some class of order continuous positive operators that behave in many instances like functionals. In fact such operators are representable as Boolean valued order continuous functionals.

4.1.1. Throughout this section X and Y are vector lattices with Y Dedekind complete. A linear operator $T : X \rightarrow Y$ is said to have the *Maharam property* or is said to be *order interval preserving* whenever $T[0, x] = [0, Tx]$ for every $0 \leq x \in X$; i. e., if for every $0 \leq x \in X$ and $0 \leq y \leq Tx$ there is some $0 \leq u \in X$ such that $Tu = y$ and $0 \leq u \leq x$. A *Maharam operator* is an order continuous positive operator whose modulus enjoys the Maharam property.

Say that a linear operator $S : X \rightarrow Y$ is *absolutely continuous* with respect to T and write $S \preceq T$ if $|S|x \in \{|T|x\}^{\perp\perp}$ for all $x \in X_+$. It can be easily seen that if $S \in \{T\}^{\perp\perp}$ then $S \preceq T$, but the converse may be false.

4.1.2. The null ideal \mathcal{N}_T of an order bounded operator $T : X \rightarrow Y$ is defined by $\mathcal{N}_T := \{x \in X : |T|(|x|) = 0\}$. Observe that \mathcal{N}_T is indeed an ideal in X . The disjoint complement of \mathcal{N}_T is referred to as the *carrier* of T and is denoted by \mathcal{C}_T , so that $\mathcal{C}_T := \mathcal{N}_T^\perp$. An operator T is called *strictly positive* whenever $0 < x \in X$ implies $0 < |T|(x)$. Clearly, $|T|$ is strictly positive on \mathcal{C}_T . Sometimes we find it convenient to denote $X_T := \mathcal{C}_T$ and $Y_T := (\text{Im } T)^{\perp\perp}$.

4.1.3. As an examples of Maharam operators, we consider conditional expectation and Bochner integration. Take a probability space (Q, Σ, μ) and let Σ_0 and μ_0 be a σ -subalgebra of Σ and the restriction of μ to Σ_0 . The conditional expectation operator $\mathcal{E}(\cdot, \Sigma_0)$ is a Maharam operator from $L^1(Q, \Sigma, \mu)$ onto $L^1(Q, \Sigma_0, \mu_0)$. The restriction of $\mathcal{E}(\cdot, \Sigma_0)$ to $L^p(Q, \Sigma, \mu)$ is also a Maharam operator from $L^p(Q, \Sigma, \mu)$ to $L^p(Q, \Sigma_0, \mu_0)$. These facts are immediate in view of the simple properties of conditional expectation.

Let (Q, Σ, μ) be a probability space, and let Y be a Banach lattice. Consider the space $X := L^1(Q, \Sigma, \mu, F)$ of Bochner integrable Y -valued functions, and let $T : E \rightarrow F$ denote the Bochner integral $Tf := \int_Q f d\mu$. If the Banach lattice Y has order continuous norm (in this case Y is order complete) then X is a Dedekind complete vector lattice under the natural order $f \geq 0 \Leftrightarrow f(t) \geq 0$ for almost all $t \in Q$) and T is a Maharam operator. See more examples in [41, 42].

4.1.4. A positive operator $T : X \rightarrow Y$ is said to have the *Levi property* if $\sup x_\alpha$ exists in X for every increasing net $(x_\alpha) \subset X_+$, provided that the net (Tx_α) is order bounded in Y . For an order bounded order continuous operator T from X to Y denote by $\mathcal{D}_m(T)$ the largest ideal of the universal completion X^u onto which we may extend the operator T by order continuity. For a positive order continuous operator T we have $X = \mathcal{D}_m(T)$ if and only if T has the Levi property.

The following theorem describes an important property of Maharam operators, enabling us to embed them into an appropriate Boolean-valued universe as order continuous functionals.

4.1.5. Theorem. *Let X and Y be some vector lattices with Y having the projection property and let T be a Maharam operator from X to Y . Then there exist an order closed subalgebra \mathcal{B} of $\mathbb{B}(X_T)$ consisting of projection bands and a Boolean isomorphism h from $\mathbb{B}(Y_T)$ onto \mathcal{B} such that $T(h(L)) \subset L$ for all $L \in \mathbb{B}(Y_T)$.*

The Boolean algebra of projections \mathcal{B} in Theorem 4.1.5 as well as the corresponding Boolean algebra of bands admits a simple description. For $L \in \mathbb{B}(Y_T)$ denote by $h(L)$ the band in $\mathbb{B}(X_T)$ corresponding to the band projection $h([K])$.

4.1.6. *For a band $K \in \mathbb{B}(X_T)$ the following are equivalent:*

- (1) $Tu = Tv$ and $u \in K$ imply $v \in K$ for all $u, v \in X_+$.
- (2) $T(K'_+) \subset T(K_+)^{\perp\perp}$ implies $K' \subset K$ for all $K' \in \mathbb{B}(X_T)$.
- (3) $K = h(L)$ for some $L \in \mathbb{B}(Y)$.

A band $K \in \mathbb{B}(X_T)$ (as well as the corresponding band projection $[K] \in \mathbb{P}(X_T)$) is said to be *T -saturated* if one of (and then all) the conditions 4.1.6 (1–3) is fulfilled.

The following can be deduce from 4.1.6 by the Freudenthal Spectral Theorem.

4.1.7. *If X and Y are Dedekind complete vector lattices and T is a Maharam operator from X to Y , then there exists an f -module structure on X over an f -algebra $\mathcal{Z}(Y)$ such that an order bounded operator $S : X \rightarrow Y$ is absolutely continuous with respect to T if and only if S is $\mathcal{Z}(Y)$ -linear.*

We now state the main result of the section.

4.1.8. Theorem. Let X be a vector lattice, $Y := \mathcal{R}\downarrow$, and let $T : X \rightarrow Y$ be a strictly positive Maharam operator with $Y = Y_T$. Then there are $\mathcal{X}, \tau \in \mathbb{V}^{(\mathbb{B})}$ satisfying the following:

(1) $\mathbb{V}^{(\mathbb{B})} \models$ “ \mathcal{X} is a Dedekind complete vector lattice and $\tau : \mathcal{X} \rightarrow \mathcal{R}$ is an strictly positive order continuous functional with the Levi property”.

(2) $\mathcal{X}\downarrow$ is a Dedekind complete vector lattice and a unitary f -module over the f -algebra $\mathcal{R}\downarrow$.

(3) $\tau\downarrow : \mathcal{X}\downarrow \rightarrow \mathcal{R}\downarrow$ is a strictly positive Maharam operator with the Levi property and an $\mathcal{R}\downarrow$ -module homomorphism.

(4) There exists a lattice isomorphism φ from X into $\mathcal{X}\downarrow$ such that $\varphi(X) \subset X^\delta \subset \mathcal{X}\downarrow \subset X^u$ is a universal completion of X and $T = \tau\downarrow \circ \varphi$.

4.1.9. The Maharam operators stems from the theory of Maharam’s “full-valued” integrals, see [64, 65, 66]. Theorem 4.1.8 was established in [38, 39]. More results, applications, and references on Maharam operators are in [41, 42]. See [46] for some extension of this theory to sublinear and convex operators.

4.2. Representation of Order Continuous Operators

Theorem 4.1.8 enables us to state that every fact on order continuous functionals ought to have a parallel variant for a Maharam operators which may be proved by the Boolean valued machinery. The aim of this section is to prove an operator version of the following result.

4.2.1. Theorem. Let X be a vector lattice and X_n^\sim separates the points of X . Then there exist order dense ideals L and X' in X^u and a linear functional $\tau : L \rightarrow \mathbb{R}$ such that

- (1) $X' = \{x' \in X' : xx' \in L \text{ for all } x \in X\}$.
- (2) τ is strictly positive, o -continuous, and has the Levi property.
- (3) For every $\sigma \in X_n^\sim$ there exists a unique $x' \in X'$ such that

$$\sigma(x) = \tau(x \cdot x') \quad (x \in X).$$

- (4) The map $\sigma \mapsto x'$ is a lattice isomorphism of X_n^\sim onto X' .

4.2.2. To translate Theorem 4.10.1 into a result on operators we need some preparations. Let X and Y be f -modules over an f -algebra A . A linear operator $T : X \rightarrow Y$ is called A -linear if $T(ax) = aTx$ for all $x \in X$ and $a \in A$. Denote by $L^A(X, Y)$ the set of all order bounded A -linear operators from X to Y and put $L_n^A(X, Y) := L^A(X, Y) \cap L_n^\sim(X, Y)$.

Say that a set $\mathcal{T} \subset L^\sim(X, Y)$ separates the points of X whenever, given nonzero $x \in X$, there exists $T \in \mathcal{T}$ such that $Tx \neq 0$. In the case of a Dedekind complete Y and the sublattice $\mathcal{T} \subset L^\sim(X, Y)$ this is equivalent to saying that for every nonzero $x \in X_+$ there is a positive operator $T \in \mathcal{T}$ with $Tx \neq 0$.

4.2.3. Given a real vector lattice \mathcal{X} within $\mathbb{V}^{(\mathbb{B})}$, denote by \mathcal{X}^\sim and \mathcal{X}_n^\sim the internal vector lattices of order bounded and order continuous functionals on \mathcal{X} , respectively. More precisely, $\llbracket \sigma \in \mathcal{X}^\sim \rrbracket = 1$ and $\llbracket \sigma \in \mathcal{X}_n^\sim \rrbracket = 1$ mean

that $\llbracket \sigma : \mathcal{X} \rightarrow \mathcal{R} \text{ is an order bounded functional} \rrbracket = 1$ and $\llbracket \sigma : \mathcal{X} \rightarrow \mathcal{R} \text{ is an order continuous functional} \rrbracket = 1$, respectively. Put $X := \mathcal{X} \downarrow$ and $A := \mathcal{R} \downarrow$.

4.2.4. Theorem. *The mapping assigning to each $\sigma \in \mathcal{X}^\sim \downarrow$ its descent $S := \sigma \downarrow$ is a lattice isomorphism of $\mathcal{X}^\sim \downarrow$ and $\mathcal{X}_n^\sim \downarrow$ onto $L^A(X, \mathcal{R} \downarrow)$ and $L_n^A(X, \mathcal{R} \downarrow)$, respectively. Moreover,*

$$\llbracket \mathcal{X}^\sim \text{ (resp. } \mathcal{X}_n^\sim) \text{ separates the points of } \mathcal{X} \rrbracket = 1$$

if and only if $L^A(X, \mathcal{R} \downarrow)$ (resp. $L_n^A(X, \mathcal{R} \downarrow)$) separates the points of X .

4.2.5 By the Gordon Theorem we may assume also that $Y^u = \mathcal{R} \downarrow$. Of course, in this event we can identify A^u with Y^u . In view of Theorem 1.3.7 there exists a real Dedekind complete vector lattice \mathcal{X} within $\mathbb{V}^{(\mathbb{B})}$ with $\mathbb{B} = \mathbb{P}(Y)$ such that $\mathcal{X} \downarrow$ is an f -module over A^u , and there is an f -module isomorphism h from X to $\mathcal{X} \downarrow$ satisfying $\mathcal{X} \downarrow = \text{mix}(h(X))$. In virtue of 4.2.4 \mathcal{X}_n^\sim separates the points of \mathcal{X} . The Transfer Principle tells us that Theorem 4.2.1 is true within $\mathbb{V}^{(\mathbb{B})}$, so that there exist an order dense ideal \mathcal{L} in \mathcal{X}^u and a strictly positive linear functional $\tau : \mathcal{L} \rightarrow \mathcal{R}$ with the Levi property such that the order ideal $\mathcal{X}' = \{x' \in \mathcal{X}^u : x' \mathcal{X} \subset \mathcal{L}\}$ is lattice isomorphic to \mathcal{X}_n^\sim ; moreover, the isomorphism is implemented by assigning the functional $\sigma_{x'} \in \mathcal{X}_n^\sim$ to $x' \in \mathcal{X}'$ by $\sigma_{x'}(x) = \tau(xx')$ ($x \in \mathcal{X}$).

4.2.6 Put $\hat{X} := \mathcal{X} \downarrow$, $\hat{L} := \mathcal{L} \downarrow$, $\hat{T} := \tau \downarrow$, and $\hat{X}' := \mathcal{X}' \downarrow$. By Theorem 2.11.9 we can identify the universally complete vector lattices X^u , \hat{X}^u , and $\mathcal{X}^u \downarrow$ as well as X with a laterally dense sublattice in \hat{X} . Then \hat{L} is an order dense ideal in \hat{X}^u and an f -module over A^u , while $\hat{T} : \hat{L} \rightarrow Y^u$ is a strictly positive Maharam operator with the Levi property. Since the multiplication in X^u is the descent of the internal multiplication in \mathcal{X}^u , we have the representation $\hat{X}' = \{x' \in X^u : x' \hat{X} \subset \hat{L}\}$. Moreover, \hat{X}' is f -module isomorphic to $L_n^A(\hat{X}, Y^u)$ by assigning to $x' \in \hat{X}'$ the operator $\hat{S}_{x'} \in L_n^A(\hat{X}, Y^u)$ defined as $\hat{S}_{x'}(x) = \hat{T}(xx')$ ($x \in \hat{X}$). Now, defining

$$L := \{x \in \hat{L} : \hat{T}x \in Y\}, \quad T := \hat{T}|_L, \\ X' := \{x' \in \hat{X}' : x' X \subset L\},$$

yields that if $x' \in X'$ then $S_{x'} := \hat{S}_{x'}|_X$ is contained in $L_n^A(X, Y)$. Conversely, an arbitrary $S \in L_n^A(X, Y)$ has a representation $Sx = \hat{T}(xx')$ ($x \in X$) with some $x' \in \hat{X}'$, so that $\hat{T}(xx') \in Y$ for all $x \in X$ and hence $x' \in X'$, $xx' \in L$ for all $x \in X$, and $Sx = T(xx')$ ($x \in X$) by the above definitions.

4.2.7. Theorem. *Let X be an f -module over $A := \mathcal{Z}(Y)$ with Y being a Dedekind complete vector lattice and let $L_n^A(X, Y)$ separates the points of X . Then there exist an order dense ideal L in X^u and a strictly positive Maharam operator $T : L \rightarrow Y$ such that the order ideal $X' = \{x' \in X^u : (\forall x \in X) xx' \in L\} \subset X^u$ is lattice isomorphic to $L_n^A(X, Y)$. The isomorphism is implemented by assigning the operator $S_{x'} \in L_n^A(X, Y)$ to an element $x' \in X'$ by the formula*

$$S_{x'}(x) = \Phi(xx') \quad (x \in X).$$

If there exists a strictly positive $T_0 \in L_n^A(X, Y)$ then one can choose L and T such that $X \subset L$ and $T|_X = T_0$.

Below, in 4.2.8–4.2.10, X and Y are Dedekind complete vector lattices.

4.2.7. Hahn Decomposition Theorem. *Let $S : X \rightarrow Y$ be a Maharam operator. Then there is a band projection $\pi \in \mathbb{P}(X)$ such that $S^+ = S \circ \pi$ and $S^- = -S \circ \pi^\perp$. In particular, $|S| = S \circ (\pi - \pi^\perp)$.*

4.2.8. Nakano Theorem. *Let $T_1, T_2 : X \rightarrow Y$ be order bounded operators such that $T := |T_1| + |T_2|$ is a Maharam operator. Then T_1 and T_2 are disjoint if and only if so are their carriers; symbolically, $T_1 \perp T_2 \iff \mathcal{C}_{T_1} \perp \mathcal{C}_{T_2}$.*

4.2.9. Radon–Nikodým Theorem. *Assume that $T : X \rightarrow Y$ be a positive Maharam operator. A positive operator $S : X \rightarrow Y$ belongs to $\{T\}^{\perp\perp}$ if and only if there exists an orthomorphism $0 \leq \rho \in \text{Orth}^\infty(X)$ with $Sx = T(\rho x)$ for all $x \in \mathcal{D}(\rho)$.*

4.2.11. Theorem 4.2.1 is proved in [82, Theorem 2.1]. It can be also extracted from [81, Theorem IX.3.1] or [41, Theorem 3.4.8]. Theorem 4.2.7 was proved in [38]; also see [41]. Theorems 4.2.8–4.2.10, first obtained in [62], may be deduced easily from 4.2.7, or can be proved by the general scheme of “Boolean valued scalarization.”

4.3. Conditional Expectation Type Operators

The conditional expectation operators have many remarkable properties related to the order structure of the underlying function space. Boolean valued analysis enables us to demonstrate that some much more general class of operators shares these properties.

4.3.1. Let Z be a universally complete vector lattice with unit $\mathbb{1}$. Recall that Z is an f -algebra with multiplicative unit $\mathbb{1}$. Assume that $\Phi : L^1(\Phi) \rightarrow Y$ is a Maharam operator with the Levi property. We shall write $L^0(\Phi) := Z$ whenever $L^1(\Phi)$ is an order dense ideal in Z . Denote also by $L^\infty(\Phi)$ the order ideal in Z generated by $\mathbb{1}$. Consider an order ideal $X \subset Z$ and we shall always assume that $L^\infty(\Phi) \subset X \subset L^1(\Phi)$. The associate space X' is defined as the set of all $x' \in L^0(\Phi)$ for which $xx' \in L^1(\Phi)$ for all $x \in X$. Clearly, X' is an order ideal in Z .

If (Ω, Σ, μ) is a probability space and \mathcal{X}_0 is an order closed vector sublattice of $L^\infty(\Omega, \Sigma, \mu)$ containing $\mathbb{1}_\Omega$, then there exists a σ -subalgebra Σ_0 of Σ such that $\mathcal{X}_0 = L^\infty(\Omega, \Sigma_0, \mu_0)$, with $\mu_0 = \mu|_{\mathcal{X}_0}$; cp. [19, Lemma 2.2].

Interpreting this fact and the properties of conditional expectation in a Boolean value model yields the following result.

4.3.2. Theorem. *Let $\Phi : L^1(\Phi) \rightarrow Y$ be a strictly positive Maharam operator with $Y = Y_\Phi$ and let Z_0 be an order closed sublattice in $L^0(\Phi)$. If $\mathbb{1} \in X_0 := L^1(\Phi) \cap Z_0$ and the restriction $\Phi_0 := \Phi|_{X_0}$ has the Maharam property then $X_0 = L^1(\Phi_0)$ and there exists an operator $E(\cdot|Z_0)$ from $L^1(\Phi)$ onto $L^1(\Phi_0)$ such that*

- (1) $E(\cdot|Z_0)$ is an order continuous positive linear projection.
- (2) $E(\cdot|Z_0)$ commutes with all saturated projections, i. e. $E(h(\pi)x|Z_0) =$

$h(\pi)E(x|Z_0)$ for all $\pi \in \mathbb{P}_\Phi(X)$ and $x \in L^1(\Phi)$.

(3) $\Phi(xy) = \Phi(yE(x|Z_0))$ for all $x \in L^1(\Phi)$ and $y \in L^\infty(\Phi_0)$.

(4) $\Phi_0(|E(x|Z_0)|) \leq \Phi(|x|)$ for all $x \in L^1(\Phi)$.

(5) $E(\cdot|Z_0)$ satisfies the averaging identity, i.e. $E(vE(x|Z_0)|Z_0) = E(v|Z_0)E(x|Z_0)$ for all $x \in L^1(\Phi)$ and $v \in L^\infty(\Phi)$.

4.3.3. We will call the operator $E(\cdot|Z_0)$ defined by Theorem 4.10.4 the *conditional expectation operator with respect to Z_0* . Take $w \in X'$ and observe that $E(wx|Z_0) \in L^1(\Phi_0)$ is well defined for all $x \in X$. If, moreover, $E(wx|Z_0) \in X$ for every $x \in X$ then we can define a linear operator $T : X \rightarrow X$ by putting $Tx = E(wx|Z_0)$ ($x \in X$). Clearly, T is order bounded and order continuous. Furthermore, for all $x \in X_+$ we have

$$T^+x = \mathcal{E}(w^+x|Z_0), \quad T^-x = \mathcal{E}(w^-x|Z_0), \quad |T|x = \mathcal{E}(|w|x|Z_0).$$

In particular, T is positive if and only if so is w . Putting $x := wx$ and $y := \mathbb{1}$ in 4.10.3(3), we get $\Phi(wx) = \Phi(wx\mathbb{1}) = \Phi(\mathcal{E}(wx|Z_0)) = \Phi(Tx)$ for all $x \in X$. Now, x can be chosen to be a component of $\mathbb{1}$ with $wx = w^+$ or $wx = w^-$, so that $T = 0$ implies $\Phi(w^+) = 0$ and $\Phi(w^-) = 0$, since Φ is strictly positive. Thus $w \in X'$ is uniquely determined by T .

Say that T satisfies the *averaging identity*, if $T(y \cdot Tx) = Ty \cdot Tx$ for all $x \in X$ and $y \in L^\infty(\Phi)$. Now we present two well-known results. By $\mathcal{E}(\cdot|\Sigma_0)$ we denote the conditional expectation operator with respect to σ -algebra Σ_0 .

4.3.4. Theorem. Let (Ω, Σ, μ) be probability space and \mathcal{X} be an order ideal in $L^1(\Omega, \Sigma, \mu)$ containing $L^\infty(\Omega, \Sigma, \mu)$. For a linear operator \mathcal{T} on \mathcal{X} the following are equivalent:

(1) \mathcal{T} is order continuous, satisfies the averaging identity, and leaves invariant the subspace $L^\infty(\Omega, \Sigma, \mu)$.

(2) There exist $w \in \mathcal{X}'$ and a sub- σ -algebra Σ_0 of Σ such that $\mathcal{T}x = \mathcal{E}(wx|\Sigma_0)$ for all $x \in \mathcal{X}$.

The following two results can be proved by interpreting Theorems 4.3.4 and 4.3.5 in a Boolean valued model.

4.3.5. Theorem. For a subspace \mathcal{X} of $L^1(\Omega, \Sigma, \mu)$ the following are equivalent:

(1) \mathcal{X} is the range of a positive contractive projection.

(2) \mathcal{X} is a closed vector sublattice of $L^1(\Omega, \Sigma, \mu)$.

(3) There exists a lattice isometry from some $L^1(\Omega', \Sigma', \mu')$ space onto \mathcal{X} .

4.3.6. Theorem. Let $\Phi : L^1(\Phi) \rightarrow Y$ be a strictly positive Maharam operator and X be an order dense ideal in $L^1(\Phi)$ containing $L^\infty(\Phi)$. For a linear operator T on X the following are equivalent:

(1) T is order continuous, satisfies the averaging identity, leaves invariant the subspace $L^\infty(\Phi)$, and commutes with all Φ -saturated projections.

(2) There exist $w \in X'$ and an order closed sublattice Z_0 in $L^0(\Phi)$ containing a unit element $\mathbb{1}$ of $L^1(\Phi)$ such that the restriction of Φ onto $L^1(\Phi) \cap Z_0$ has the Maharam property and $Tx = \mathcal{E}(wx|Z_0)$ for all $x \in X$.

4.3.7. Theorem. For each subspace X_0 of $L^1(\Phi)$ the following statements are equivalent:

- (1) X is the range of a positive Φ -contractive projection.
- (2) X is a closed vector sublattice of $L^1(\Phi)$ invariant under all Φ -saturated projections.

(3) There exists a Maharam operator $\Psi : L^1(\Psi) \rightarrow Y$ and a lattice isomorphism h from $L^1(\Psi)$ onto X such that $\Phi(|Tx|) = \Psi(|x|)$ for all $x \in L^1(\Psi)$.

4.3.8. Theorems 4.3.4 and 4.3.6 can be found in [19, Proposition 3.1] and [20, Lemma 1], respectively. Theorems 4.3.6 and 4.3.7 are published for the first time.

4.4. Maharam Extension

Thus, the general properties of Maharam operators can be deduced from the corresponding facts about functionals with the help of Theorem 4.1.8. Nevertheless, these methods may be also useful in studying arbitrary regular operators.

4.4.1. Suppose that X is a vector lattice over a dense subfield $\mathbb{F} \subset \mathbb{R}$ and $\varphi : X \rightarrow \mathbb{R}$ is a strictly positive \mathbb{F} -linear functional. There exist a Dedekind complete vector lattice X^φ containing X and a strictly positive order continuous linear functional $\bar{\varphi} : X^\varphi \rightarrow \mathbb{R}$ with the Levi property extending φ such that for every $x \in X^\varphi$ there is a sequence (x_n) in X with $\lim_{n \rightarrow \infty} \bar{\varphi}(|x - x_n|) = 0$.

\triangleleft Denote $d(x, y) := \varphi(|x - y|)$ and note that (X, d) is a metric space. Let X^φ the completion of the metric space (X, d) and $\bar{\varphi}$ is an extension of φ to X^φ by continuity. It is not difficult to ensure that X^φ is a Banach lattice with an additive norm $\|\cdot\|^\varphi := \bar{\varphi}(|\cdot|)$ containing X as a norm dense \mathbb{F} -linear sublattice. Thus, $\bar{\varphi}$ is a strictly positive order continuous linear functional on X^φ with the Levi Property. \triangleright

4.4.2. Denote $L^1(\varphi) := X^\varphi$ and let \bar{X} stands for the order ideal in $L^1(\varphi)$ generated by X . Then $(L^1(\varphi), \|\cdot\|^\varphi)$ is an AL -space and \bar{X} is a Dedekind complete vector lattice. Moreover, X is norm dense in $L^1(\varphi)$ and hence in \bar{X} .

Given a nonempty subset U of a lattice L , we denote by U^\uparrow (resp. U^\downarrow) the set of elements $x \in L$ representable in the form $x = \sup(A)$ (resp. $x = \inf(A)$), where A is an upward (resp. downward) directed subset of U . Moreover, we set $U^{\uparrow\downarrow} := (U^\uparrow)^\downarrow$ etc. If in the above definition A is countable, then we write U^1 , U^\downarrow , and $U^{1\downarrow}$ instead of U^\uparrow , U^\downarrow , and $U^{\uparrow\downarrow}$. Recall that for the Dedekind completion X^δ we have $X^\delta = X^\uparrow = X^\downarrow$.

4.4.3. The identities $\bar{X} = X^{11} = X^{1\downarrow}$ and $L^1(\varphi) = X^{11} = X^{1\downarrow}$ hold with both $(\cdot)^{11}$ and $(\cdot)^{1\downarrow}$ taken in \bar{X} and $L^1(\varphi)$, respectively.

Translating 4.4.1 and 4.4.2 by means of Boolean valued “scalarization” leads to the following result.

4.4.4. Theorem. Let X and Y be vector lattices with Y Dedekind complete and T a strictly positive linear operator from X to Y . There exist a Dedekind complete vector lattice \bar{X} and a strictly positive Maharam operator $\bar{T} : \bar{X} \rightarrow Y$ satisfying the conditions:

(1) There exist a lattice homomorphism $\iota : X \rightarrow \bar{X}$ and an f -algebra homomorphism $\theta : \mathcal{Z}(Y) \rightarrow \mathcal{Z}(\bar{X})$ such that

$$\alpha T x = \bar{T}(\theta(\alpha)\iota(x)) \quad (x \in X, \alpha \in \mathcal{Z}(Y)). \quad (2)$$

(2) $\iota(X)$ is a majorizing sublattice in \bar{X} and $\theta(\mathcal{Z}(Y))$ is an order closed sublattice and subring of $\mathcal{Z}(\bar{X})$.

(3) The representation $\bar{X} = (X \odot \mathcal{Z}(Y))^{\uparrow\downarrow}$ holds, where $X \odot \mathcal{Z}(Y)$ is a subspace of \bar{X} consisting of all finite sums $\sum_{k=1}^n \theta(\alpha_k)\iota(x_k)$ with $x_1, \dots, x_n \in X$ and $\alpha_1, \dots, \alpha_n \in \mathcal{Z}(Y)$.

4.4.5. The pair (\bar{X}, \bar{T}) (or \bar{T} for short) is called a *Maharam extension* of T if it satisfies 4.4.4(1–3). The pair (\bar{X}, ι) is also called a *Maharam extension space* for T . Two Maharam extensions T_1 and T_2 of T with the respective Maharam extension spaces (X_1, ι_1) and (X_2, ι_2) are said to be *isomorphic* if there exists a lattice isomorphism h of X_1 onto X_2 such that $T_1 = T_2 \circ h$ and $\iota_2 = h \circ \iota_1$. It is not difficult to ensure that a Maharam extension is unique up to isomorphism.

4.4.6. Let X and Y be vector lattices with Y Dedekind complete, $T : X \rightarrow Y$ a strictly positive operator and (\bar{X}, \bar{T}) a Maharam extension of T . Consider a universal completion \bar{X}^u of \bar{X} with a fixed f -algebra structure. Let $L_1(\Phi)$ be the greatest order dense ideal in \bar{X}^u , onto which \bar{T} can be extended by order continuity. In more details,

$$\begin{aligned} L^1(T) &:= \{x \in \bar{X}^u : \bar{T}([0, |x|] \cap \bar{X}) \text{ is order bounded in } Y\}, \\ \hat{T}x &:= \sup\{\bar{T}u : u \in \bar{X}, 0 \leq u \leq x\} \quad (x \in L^1(T)_+), \\ \hat{T}x &= \hat{T}x^+ - \hat{T}x^- \quad (x \in L^1(T)). \end{aligned}$$

Define an Y -valued norm $\|\cdot\|$ on $L^1(T)$ by $\|u\| := \hat{T}(|u|)$. In terminology of lattice normed spaces $(L^1(T), \|\cdot\|)$ is a *Banach–Kantorovich lattice*, see [41, Chapter 2]. In particular, $\|au\| = |a|\|u\|$ ($a \in \mathcal{Z}(Y)$, $u \in L^1(T)$).

4.4.7. Theorem. For every operator $S \in \{T\}^{\perp\perp}$ there is a unique element $z = z_T \in \bar{X}^u$ satisfying

$$Sx = \hat{T}(z \cdot \iota(x)) \quad (x \in X).$$

The correspondence $T \mapsto z_T$ establishes a lattice isomorphism between the band $\{T\}^{\perp\perp}$ and the order dense ideal in \bar{X}^u defined by

$$\{z \in \bar{X}^u : z \cdot \iota(X) \subset L_1(T)\}.$$

◁ This result is a variant of the Radon–Nikodým Theorem for positive operators and may be obtained as a combination of Theorems 4.2.10 and 4.4.4 or proved by means of Boolean valued “scalarization”. ▷

4.4.8. Maharam extension stems from the corresponding extension result given by D. Maharam for F -integrals [64, 65, 66]. For operators in Dedekind complete vector lattices this construction was performed in [8, 9] by three different ways. One of them, based upon the imbedding $x \mapsto \bar{x}$ of a vector lattice

X into $L^\sim((L^\sim(X, Y), Y)$ defined as $\bar{x}(T) := Tx$ ($T \in L^\sim(X, Y)$), was independently discovered in [61]. The main difference is that in [61] the Maharam extension was constructed for an arbitrary collection of order bounded operators. For some further properties of Maharam extension see in [41] and [61].

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